

# The tropical momentum map: a classification of toric log symplectic manifolds

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We give a generalization of toric symplectic geometry to Poisson manifolds which are symplectic away from a collection of hypersurfaces forming a normal crossing configuration. We introduce the tropical momentum map, which takes values in a generalization of affine space called a log affine manifold. Using this momentum map, we obtain a complete classification of such manifolds in terms of decorated log affine polytopes, hence extending the classification of symplectic toric manifolds achieved by Atiyah, Guillemin-Sternberg, Kostant, and Delzant.

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# 1 Introduction

Toric symplectic geometry was revolutionized in the 1970s and 1980s by Atiyah, Guillemin-Sternberg, Kostant and Delzant [1, 8, 14, 5], who essentially proved that toric symplectic manifolds are encoded combinatorially by a rational polytope, which is the image of the classical momentum map associated to the toric action (this is usually referred to as the *Delzant correspondence*). In the present paper, we present a generalization of toric symplectic geometry to a class of Poisson manifolds, called *log symplectic manifolds*, which are generically symplectic but degenerate along a normal crossing configuration of smooth hypersurfaces. We give a classification analogous to the one given by the aforementioned authors, but incorporating new invariants coming from the degeneracy loci.

The study of toric log symplectic manifolds was initiated in the paper of Guillemin, Miranda, Pires and Scott [10], who developed an extension of the Delzant correspondence in the case where the degeneracy locus of the Poisson structure is a smooth hypersurface.

Degeneracy loci for Poisson structures of interest are, however, often highly singular. In this paper, we consider the mildest possible class of singularities: normal crossing hypersurfaces. This class is particularly interesting, as it is simple enough to allow for the presence of toric symmetries, yet is extremely rich in examples and in combinatorial structure, in comparison to the nonsingular case.

To make progress on the normal crossing case, it is necessary to reevaluate the nature of the momentum map itself, and to investigate the geometric structure present on its image, which is the direct analog of the rational polytope of Atiyah et al. mentioned above. We show that a global *tropical moment map* may be defined, but the geometry and topology of its codomain is significantly more involved than in the case where the degeneracy locus is smooth (or empty). In the classical case, the momentum image is contractible, while in our case there is no a priori constraint on its topology. In Example 5.21, for instance, we obtain a momentum image diffeomorphic to a surface of genus 1 with boundary, corresponding to a log symplectic 4-manifold which, interestingly, admits no symplectic structure due to the vanishing of its Seiberg-Witten invariants.

In order to use the tropical moment map effectively to achieve a classification, we introduce several ideas from other fields, such as the Mazzeo-Melrose decomposition for manifolds with corners, the concept of free divisors in algebraic geometry, and the notion of tropicalization from tropical geometry. Indeed, we assemble the possible codomains for tropical moment maps from elementary pieces called *tropical domains*, which are partial compactifications of affine spaces closely related to the extended tropicalizations of toric varieties defined by Kajiwara [12] and Payne [19]. In fact, our work was initially motivated by the problem of defining moment maps for Caine's examples of real Poisson structures on complex toric varieties [2]. Such a variety may be blown up along its circle fixed point loci to produce a log symplectic manifold with corners, and in this way we may identify its tropical moment map with its tropicalization morphism.

By carefully selecting these tools, we are able to reach the classification with a minimum of technical fuss. This is a simplified restatement of our main result, which appears with full generality and detail as Theorem 6.3 in Section 6:

**Theorem** (Classification of Hamiltonian toric log symplectic manifolds). *There is a one-to-one correspondence between equivariant isomorphism classes of oriented compact connected toric Hamiltonian log symplectic  $2n$ -manifolds and equivalence classes of pairs  $(\Delta, M)$ , where  $\Delta$  is a compact convex log affine polytope of dimension  $n$  satisfying the Delzant condition and  $M$  is a principal  $n$ -torus bundle over  $\Delta$  with vanishing toric log obstruction class.*

Placing our work in the larger context of the study of integrable systems, we provide a classification of a large family of toric integrable systems in which the base of the system is a manifold whose integral affine structure is allowed to degenerate in a controlled manner along a stratification. The results of this paper provide further evidence that tools from Lie algebroid theory may be fruitfully exploited to understand and classify the singular behaviour of the affine structure on the base of integrable systems, and therefore to further our ability to classify integrable systems at large.

The structure of the paper is as follows: In Section 3, we define log affine manifolds and provide a method for constructing a great variety of examples by welding together tropical domains. In Section 4, we define toric log symplectic manifolds and classify those with principal torus actions. In section 5, we describe the analogue of the Delzant polytope and provide several examples, including the key example 5.21. Finally, in Section 6, we establish the correspondence between log affine polytopes and toric log symplectic manifolds.

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## 2 Notation and conventions

- i) *Manifolds with corners*: Throughout the paper, manifolds have boundary and corners. We use the definition in which each local chart has codomain given by an arbitrary union or intersection of coordinate half-spaces  $\pm x_i \geq 0$ , in standard coordinates  $(x_1, \dots, x_n)$  on  $\mathbb{R}^n$ .
- ii) *Lie algebroids*: A Lie algebroid is a vector bundle  $A$  over the manifold  $X$ , together with a Lie bracket  $[\cdot, \cdot]$  on its smooth sections and a morphism of vector bundles  $\varrho : A \rightarrow TX$ , called the *anchor*, which is bracket-preserving and satisfies the Leibniz rule

$$[a, fb] = f[a, b] + (\varrho(a)f)b,$$

for all  $a, b$  sections of  $A$  and  $f \in C^\infty(X, \mathbb{R})$ .

- iii) *Action Lie algebroid*: Given an infinitesimal action of a Lie algebra  $\mathfrak{g}$  on a manifold  $X$ , i.e., a Lie algebra homomorphism  $\rho : \mathfrak{g} \rightarrow C^\infty(X, TX)$ , the *action Lie algebroid*  $\mathfrak{g} \ltimes_\rho X$  is the trivial bundle  $\mathfrak{g} \times X \rightarrow X$ , equipped with anchor map  $\varrho$  defined by  $\varrho(a) = (x, \rho(a)(x))$  and Lie bracket on sections given by the unique extension of the Lie bracket on constant sections which satisfies the Leibniz rule.
- iv) *Free divisors*: A *free divisor* is defined to be a union of smooth closed hypersurfaces (i.e., real codimension one submanifolds) with the property that the sheaf of vector fields tangent to all hypersurfaces is locally free. Given a free divisor  $D$  on the  $n$ -manifold  $X$ , we let  $TX(-\log D)$  denote the rank  $n$  vector bundle defined by the above sheaf. It is a Lie algebroid, with Lie bracket inherited from the Lie bracket of vector fields and anchor map defined by the natural inclusion of sheaves. Its dual bundle is denoted by  $T^*X(\log D)$ , and the Lie algebroid de Rham complex is usually written  $(\Omega^\bullet(X, \log D), d)$ , and is called the *logarithmic de Rham complex* (See Appendix A.4).
- v) *Normal crossing divisors*: A convenient subclass of free divisors are those with only simple normal crossing singularities, meaning that each point  $x \in D$ , there is a chart  $(U, \varphi)$  of  $X$ ,  $x \in U$ , such that  $\varphi(D)$  is a union of a subset of the coordinate hyperplanes in  $\mathbb{R}^n$  intersected with  $\varphi(D)$ . The *smooth locus* of  $D$  is defined to be the set of points in  $D$  which lie only on a single hypersurface of  $D$ . The *singular locus* of  $D$  is the complement in  $D$  of the smooth locus.

### 3 Log affine manifolds

**3.1** A toric Hamiltonian symplectic manifold is equipped with a momentum map to the dual of the Lie algebra of the torus: the affine space  $\mathfrak{t}^*$ . When generalizing the theory to log symplectic forms, one is led to replace the codomain  $\mathfrak{t}^*$  by a *log affine manifold*, possibly with boundary and corners:

**Definition 3.2.** A real log affine  $n$ -manifold is a manifold  $X$  of real dimension  $n$ , equipped with a free divisor  $D$  and an isomorphism  $\xi$  of Lie algebroids between  $TX(-\log D)$  and an abelian action algebroid, i.e.

$$TX(-\log D) \xrightarrow{\xi} \mathbb{R}^n \ltimes_{\rho} X, \quad (1)$$

where  $\rho : \mathbb{R}^n \rightarrow C^\infty(X, TX)$  is a Lie algebra homomorphism. We call a log affine manifold complete when the infinitesimal action of  $\mathbb{R}^n$  integrates to a group action of  $\mathbb{R}^n$  on  $X$ . The isomorphism (1) may be viewed equivalently as a closed logarithmic 1-form

$$\xi \in \Omega^1(X, \log D) \otimes \mathbb{R}^n. \quad (2)$$

**Remark 3.3.** If  $X$  has boundary or corners and if  $D$  is a normal crossing divisor, then in order to be complete log affine, the divisor  $D$  must include the normal crossing boundary components of  $X$ .

**Remark 3.4.** Saito's criterion [20] provides a convenient way to characterize log affine manifolds. A simple consequence of the criterion is the following: if  $(X_1, \dots, X_n)$  are commuting vector fields on the  $n$ -manifold  $X$  which are tangent to a normal crossings divisor  $D$ , and if their determinant

$$X_1 \wedge \dots \wedge X_n \in C^\infty(X, \wedge^n TX) \quad (3)$$

vanishes precisely on  $D$  and trasversely on its smooth locus, then  $TX(-\log D)$  is an action Lie algebroid globally generated by the  $n$  sections  $(X_1, \dots, X_n)$ .

**3.5** Assuming that  $D$  is a normal crossing divisor, by the Mazzeo-Melrose decomposition (A.4), we have the natural isomorphism

$$H^1(X, \log D) \otimes \mathbb{R}^n \cong (H^1(X) \oplus \sum_i H^0(D_i)) \otimes \mathbb{R}^n.$$

As a result, the logarithmic 1-form (2) defines a special class in first cohomology.

**Definition 3.6.** The affine monodromy of the log affine manifold  $(X, D, \xi)$  (where  $D$  has normal crossing type) is defined to be the component of  $[\xi]$  in  $H^1(X) \otimes \mathbb{R}^n$ .

The log affine manifolds we consider in this paper will always be assumed to have trivial affine monodromy, and the gluing construction of log affine manifolds presented in this section is designed to produce examples with trivial affine monodromy.

### 3.1 Tropical domains

**3.7** Let  $X$  be an affine space for the real vector space  $U$  of dimension  $n$ , and fix a nonzero vector  $\alpha \in U$ . We now define a real analogue of the symplectic cut construction which partially compactifies  $X$  to a manifold with boundary; the boundary is a copy of  $X/\langle\alpha\rangle$  located “at infinity” in the  $-\alpha$  direction.

**Definition 3.8.** *Let  $(X, U, \alpha)$  be as above. Then we define the manifold with boundary*

$$\overline{X}_\alpha = (X \times \mathbb{R}_+) / \mathbb{R}, \quad (4)$$

where  $\mathbb{R}_+ = [0, \infty)$  and  $s \in \mathbb{R}$  acts via the (free and proper) anti-diagonal action

$$s \cdot (x, \lambda) = (x - s\alpha, e^s \lambda). \quad (5)$$

$X$  is then included in  $\overline{X}_\alpha$  via  $x \mapsto [(x, 1)]$ .

The choice of an affine hyperplane  $H \subset X$  transverse to  $\alpha$  defines a slice for the action in (4), identifying  $\overline{X}_\alpha$  with the half-space  $H_+ = H + \mathbb{R}_+\alpha$ . The slice is given by

$$\begin{aligned} H_+ &\longrightarrow X \times [0, \infty), \\ x &\longmapsto (x^\perp, \lambda) \end{aligned} \quad (6)$$

where  $x = x^\perp + \lambda\alpha$  and  $x^\perp \in H$ . In this way we see that  $\overline{X}_\alpha$  is isomorphic to the closed half-space  $H_+$ , and so is non-canonically isomorphic to  $X/\langle\alpha\rangle \times \mathbb{R}_+$  as a manifold with boundary.

**Proposition 3.9.** *The residual action of  $U$  on  $\overline{X}_\alpha$  given by*

$$[(x, \lambda)] + u = [(x + u, \lambda)] \quad (7)$$

*renders  $\overline{X}_\alpha$  into a log affine manifold; that is,  $T\overline{X}_\alpha(-\log \partial\overline{X}_\alpha)$  is naturally isomorphic to the action algebroid  $U \ltimes \overline{X}_\alpha$  as a Lie algebroid, in a way which sends the normal Euler vector field of the boundary to the translation  $\alpha$ .*

The last statement may also be interpreted in the following way: as in Definition 3.2, the identification of  $T\overline{X}_\alpha(-\log \partial\overline{X}_\alpha)$  with  $U \ltimes \overline{X}_\alpha$  defines a logarithmic  $U$ -valued 1-form

$$\xi \in \Omega^1(\overline{X}_\alpha, \log \partial\overline{X}_\alpha) \otimes U, \quad (8)$$

and the residue of this form along the boundary  $\partial\overline{X}_\alpha$  is  $\alpha \in U$ .

**3.10** The compactification introduced in Definition 3.8 may be applied to any complete log affine manifold  $X$  in the sense of Definition 3.2, so long as the  $\mathbb{R}$ -action generated by  $\alpha$  remains free and proper.

**Proposition 3.11.** *Let  $(X, D, U)$  be a complete log affine manifold and let  $\alpha \in U$  generate a free and proper action of  $\mathbb{R}$  on  $X$ . Then the partial compactification  $\overline{X}_\alpha$  of  $X$  defined by (4) is also a complete log affine manifold, and as in Proposition 3.9, the residue of (8) along its new boundary component is  $\alpha$ .*

**3.12** We now apply the partial compactification several times to the affine space  $X$ , iterating over a  $k$ -tuple of vectors  $(\alpha_1, \dots, \alpha_k)$  in  $U$ . For freeness and properness of each successive  $\mathbb{R}$ -action, we assume that the vectors are linearly independent. The resulting log affine manifold (now with corners) does not depend on the ordering of the  $k$ -tuple, and we have the following description of it.

**Proposition 3.13.** *Let  $X$  be an affine space for the real vector space  $U$  of dimension  $n$ , and let  $A = \{\alpha_1, \dots, \alpha_k\}$  be a set of linearly independent vectors in  $U$ . Then we define the manifold with corners*

$$\overline{X}_A = (X \times \mathbb{R}_+^k) / \mathbb{R}^k, \quad (9)$$

where  $\mathbb{R}^k$  acts via the (free and proper) anti-diagonal action

$$(s_1, \dots, s_k) \cdot (x, (\lambda_1, \dots, \lambda_k)) = (x - \sum_{i=1}^k s_i \alpha_i, (e^{s_1} \lambda_1, \dots, e^{s_k} \lambda_k)). \quad (10)$$

The residual  $U$ -action, given by

$$[(x, (\lambda_1, \dots, \lambda_k))] + u = [(x + u, (\lambda_1, \dots, \lambda_k))], \quad (11)$$

renders  $\overline{X}_A$  into a complete log affine manifold with corners.

The space  $\overline{X}_A$  constructed above is diffeomorphic to the standard corner  $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ , and we may describe each stratum of the manifold with corners as follows. The open dense stratum is a copy of the original affine space  $X$ , included in  $\overline{X}_A$  via the  $U$ -equivariant embedding  $x \mapsto [x, (1, \dots, 1)]$ . There are  $k$  codimension 1 strata: for each  $\alpha_i \in A$ , we have the equivariant embedding

$$\begin{aligned} X / \langle \alpha_i \rangle &\longrightarrow \overline{X}_A \\ x &\longmapsto [(\tilde{x}, (1, \dots, 1, 0, 1, \dots, 1))] \end{aligned} \quad (12)$$

where  $\tilde{x} \in X$  is any lift of  $x$  and 0 occurs in the  $i^{\text{th}}$  position. In a similar way, all the strata are naturally identified with the quotient affine spaces  $X / \langle A' \rangle$  corresponding to the subsets  $A' \subset A$ .

As before, the identification of algebroids defines a logarithmic  $U$ -valued 1-form

$$\xi \in \Omega^1(\overline{X}_A, \log \partial \overline{X}_A) \otimes U, \quad (13)$$

which has residue along the boundary stratum  $X / \langle \alpha_i \rangle$  given by  $\alpha_i \in U$ .

**3.14** The compactification in Proposition 3.13 may be described informally as attaching a set of  $k$  hyperplanes,  $\binom{k}{2}$  codimension 2 planes, etc. to an  $n$ -dimensional affine space at infinity, in the configuration of a single  $k$ -corner. We now explain how several such corner attachments can be made to a single affine space. While it is possible to describe multiple corner attachments using a quotient construction as above, we find it simpler to use a gluing approach, as follows.

Let  $A_0 \subset U$  be a set of linearly independent vectors in  $U$ . Suppose that we have two enlargements  $A_0 \subset A_-$  and  $A_0 \subset A_+$ , where each of  $A_-, A_+$  is a set of linearly independent vectors in  $U$ , and the convex hulls of  $A_-, A_+$  intersect precisely in the convex hull of  $A_0$ . According to the procedure described above, each of  $A_0, A_-, A_+$  determines a partial compactification  $\overline{X}_{A_0}, \overline{X}_{A_-}$ , and  $\overline{X}_{A_+}$ . It is clear from the construction that we have open embeddings of log affine manifolds as shown below

$$\begin{array}{ccc} \overline{X}_{A_-} & & \overline{X}_{A_+} \\ & \nwarrow i \quad \nearrow j & \\ & \overline{X}_{A_0} & \end{array} \quad (14)$$

The gluing of  $\overline{X}_{A_-}$  to  $\overline{X}_{A_+}$  along  $\overline{X}_{A_0}$ , i.e. the fibred coproduct, is then well-defined as a log affine manifold

$$\overline{X}_{A_- \cup A_+} = (\overline{X}_{A_-})_i \amalg_j (\overline{X}_{A_+}), \quad (15)$$

and it achieves the goal of attaching both of the corners defined by  $A_-, A_+$  to the affine space  $X$ . To organize the system of coproducts necessary to combine many such partial compactifications, it is natural to borrow from toric geometry the notion of a *fan* of strictly convex cones in  $U$ .

**3.15** Let  $U$  be a real vector space of dimension  $n$ , and let  $F$  be a finite set of vectors in  $U$ . Let  $\mathcal{C}_F$  be a collection of linearly independent subsets  $A \subset F$ , each of which is closed under convex hull in the sense that the convex hull of  $A$  does not meet  $F \setminus A$ . The collection  $\mathcal{C}_F$  must have the property that if  $A \in \mathcal{C}_F$  then all subsets of  $A$  are also contained in  $\mathcal{C}_F$ . The collection  $\mathcal{C}_F$  then determines a simplicial fan  $\Sigma_F$  consisting of the set of all strictly convex cones  $\langle A \rangle_+ \subset U$ ,  $A \in \mathcal{C}_F$ , where  $\langle A \rangle_+$  denotes the convex hull of  $A$ .



Figure 1: A fan associated to  $F = \{(1, 0), (1, 1), (0, 1)\}$

For example, we may choose a fan associated to  $F = \{(1, 0), (1, 1), (0, 1)\}$  consisting of one 2-dimensional cone and three 1-dimensional cones, as shown in Figure 1.

Given the fan  $\Sigma_F$ , we now define a partial compactification of any affine space  $X$  modeled on  $U$ , by attaching to  $X$  a collection of quotient affine spaces at infinity. Namely, for each cone  $\langle A \rangle_+ \in \Sigma_F$  of dimension  $k$ , we attach the quotient space  $X/\langle A \rangle$  at infinity, creating a codimension  $k$  stratum of a log affine manifold with corners.

**Definition 3.16.** Let  $\Sigma_F$  be the simplicial fan on  $U$  defined above, and view  $\mathcal{C}_F$  as a directed graph, ordered by containment. For any affine space  $X$  modeled on



$U$ , and any containment  $A \subset A'$ , we have an open embedding  $\overline{X}_A \rightarrow \overline{X}_{A'}$ . The induced coproduct

$$\overline{X}_{\mathcal{C}_F} = \coprod_{A \in \mathcal{C}_F} \overline{X}_A \quad (16)$$

then defines a partial compactification of  $X$  to a log affine manifold with corners, which we call a tropical domain. The space  $\overline{X}_{\mathcal{C}_F}$  is compact if and only if the fan  $\Sigma_F$  is complete, i.e. its cones cover  $U$ .

**3.17** The notion of tropical domain defined above is closely related to the notion of *extended tropicalization*  $\mathbf{Trop}(\mathcal{X})$  of a complex toric variety  $\mathcal{X}$  introduced by Kajiwara [12] and Payne [19] (see also [6] for similar ideas).

More precisely, if the fan  $\Sigma_F$  is chosen to be the fan defining a complex toric variety  $\mathcal{X}$ , and the vectors in  $F$  are chosen to be primitive integral vectors, then the resulting tropical domain can be identified with the extended tropicalization of  $\mathcal{X}$ .

Indeed, it is precisely in this case that the tropical domain serves as the correct target for the tropical momentum map defined on Caine's examples [2] of canonical real Poisson structures on complex toric varieties. To fit these examples into our formalism, we blow up their circle fixed point loci to produce a toric log symplectic manifold with corners.

### 3.2 Welding of domains

In the previous section, we explained how an affine space  $X$  modeled on the real vector space  $U$  may be partially compactified to a log affine manifold with corners  $\overline{X}_{\mathcal{C}_F}$  called a *tropical domain*. We now explain how tropical domains may be *welded* together, i.e. glued along their boundaries, in such a way that the result is a log affine manifold with corners and normal crossing degeneracy divisor. The welding is defined in such a way that the affine monodromy of the result is trivial. We call these log affine manifolds *tropical welded spaces*.

#### Welding tropical domains along a matched pair of faces

**3.18** Let  $\overline{X}_{\mathcal{C}_F}$  and  $\overline{X}_{\mathcal{C}_{F'}}$  be tropical domains with interior affine space  $X$  and corresponding finite subsets  $F, F' \subset U$ . To weld these together along the codimension 1 boundary strata corresponding to  $\alpha \in F$  and  $\alpha' \in F'$ , we require the following compatibility condition.

**Definition 3.19.** For  $\alpha \in F$ , let  $\mathcal{C}_F^\alpha \subset \mathcal{C}_F$  denote the collection of sets  $A \in \mathcal{C}_F$  which contain  $\alpha$ . We say that  $\alpha \in F$  and  $\alpha' \in F'$  are a *matched pair* when  $\mathcal{C}_F^\alpha = \mathcal{C}_{F'}^{\alpha'}$  as a set of subsets of  $U$ . In particular,  $\alpha = \alpha'$  and the fans  $\Sigma_F, \Sigma_{F'}$  share any cones containing  $\alpha$ . Any  $\beta \in F$  which is present in any set in  $\mathcal{C}_F^\alpha$  is said to be *adjacent* to  $\alpha$ , and its copy in  $F'$  is denoted  $\psi_\alpha^{\alpha'}(\beta)$ .

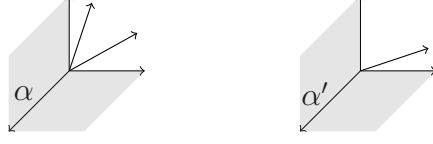


Figure 2: A matched pair  $\alpha \in F$ ,  $\alpha' \in F'$

Restricting the coproduct (16) to  $\mathcal{C}_F^\alpha \subset \mathcal{C}_F$ , we obtain

$$\mathcal{U}^\alpha = \coprod_{A \in \mathcal{C}_F^\alpha} \overline{X}_A, \quad (17)$$

defining an open neighbourhood in  $\overline{X}_{C_F}$  of the codimension 1 stratum corresponding to  $\alpha$ . The matching of  $\alpha \in F$  and  $\alpha' \in F'$  ensures that the neighbourhoods  $\mathcal{U}^\alpha$  and  $\mathcal{U}^{\alpha'}$  of the corresponding boundary strata are isomorphic.

**3.20** The welding is then defined as follows. Each of the log affine manifolds with corners  $\overline{X}_A$  constituting the coproduct (16) is canonically embedded in the log affine manifold (without corners)

$$\tilde{X}_A = (X \times \mathbb{R}^k) / \mathbb{R}^k, \quad (18)$$

where the  $\mathbb{R}^k$  action is as in (10) and the embedding of  $\overline{X}_A$  in  $\tilde{X}_A$  is via the standard inclusion of  $\mathbb{R}_+^k$  in  $\mathbb{R}^k$  as an orthant. This allows us to define a collar extension of the neighbourhood  $\mathcal{U}^\alpha$ :

$$\mathcal{U}^\alpha \subset \tilde{\mathcal{U}}^\alpha = \coprod_{A \in \mathcal{C}_F^\alpha} \tilde{X}_A. \quad (19)$$

The matching condition between  $\alpha \in F$  and  $\alpha' \in F'$  then gives rise to a canonical  $U$ -equivariant “reflection” isomorphism

$$\tilde{\mathcal{U}}^\alpha \xrightarrow{\psi^\alpha} \tilde{\mathcal{U}}^{\alpha'}, \quad (20)$$

given as follows: for each  $A \in \mathcal{C}_F^\alpha$ , we write  $A = \{\alpha, \alpha_2, \dots, \alpha_k\}$  and define  $\psi^\alpha$  on  $\tilde{X}_A \subset \tilde{\mathcal{U}}^\alpha$  via

$$[(x, (\lambda_1, \lambda_2, \dots, \lambda_k))] \xrightarrow{\psi^\alpha} [x, (-\lambda_1, \lambda_2, \dots, \lambda_k)]. \quad (21)$$

The isomorphism  $\psi^\alpha$  embeds  $\mathcal{U}^\alpha$  into  $\tilde{\mathcal{U}}^{\alpha'}$  in such a way that it intersects  $\mathcal{U}^{\alpha'}$  exactly in the closure of the codimension 1 stratum corresponding to  $\alpha$ .

**Definition 3.21.** Let  $\overline{X}_{C_F}$ ,  $\overline{X}_{C_{F'}}$  be as above and let  $\alpha \in F$ ,  $\alpha' \in F'$  be a matched pair. The welding of  $\overline{X}_{C_F}$  to  $\overline{X}_{C_{F'}}$  along  $(\alpha, \alpha')$  is defined by gluing together collar neighbourhoods of the boundary strata corresponding to  $\alpha, \alpha'$ , namely:

$$\overline{X}_{C_F} \#_{(\alpha, \alpha')} \overline{X}_{C_{F'}} = (\overline{X}_{C_F} \coprod_{\mathcal{U}^\alpha} \tilde{\mathcal{U}}^\alpha) \coprod (\overline{X}_{C_{F'}} \coprod_{\mathcal{U}^{\alpha'}} \tilde{\mathcal{U}}^{\alpha'}) / (x \sim \psi^\alpha(x)). \quad (22)$$

The result is again a complete log affine manifold with normal crossing degeneracy locus.

### General welding along a matched pair of faces

**3.22** We assume by induction that a space  $X$  has been produced by welding together a collection of tropical domains  $\{\overline{X}_{\mathcal{C}_{F_1}}, \dots, \overline{X}_{\mathcal{C}_{F_n}}\}$  according to a list of matched pairs  $\mathcal{L}$ , and we explain how to weld an additional matched pair  $(\alpha_i, \alpha_j) \in F_i \times F_j \setminus \mathcal{L}$ . Note that  $X$  need not be connected, so that this procedure includes the general welding of two tropical welded spaces.

**Definition 3.23.** Let  $(\alpha_i, \alpha_j) \in F_i \times F_j \setminus \mathcal{L}$  be a matched pair as above. The welding along  $(\alpha_i, \alpha_j)$  is locally obstructed if there exists  $\beta_i$  adjacent to  $\alpha_i$  such that either

- i)  $(\beta_i, \psi_{\alpha_i}^{\alpha_j}(\beta_i)) \in \mathcal{L}$ , that is,  $\beta_i$  is already welded to  $\psi_{\alpha_i}^{\alpha_j}(\beta_i)$ , or
- ii) There exist  $\beta_k, \alpha_l, \beta_m$  such that  $(\beta_i, \beta_k)$ ,  $(\psi_{\beta_i}^{\beta_k}(\alpha_i), \alpha_l)$  and  $(\psi_{\alpha_i}^{\alpha_j}(\beta_i), \beta_m)$  are all in  $\mathcal{L}$ ,

or either of the conditions holds with  $i, j$  exchanged.

**3.24** If a matched pair  $(\alpha_i, \alpha_j)$  is locally unobstructed, it may give rise to *coerced pairs*, namely pairs adjacent to  $\alpha_i, \alpha_j$  which should be welded in order to maintain the smoothness of the resulting log affine manifold, as shown in Figure 3.

We say that  $(\alpha_i, \alpha_j)$  is *globally unobstructed* when iteratively taking coerced pairs results only in matched pairs which are locally unobstructed.

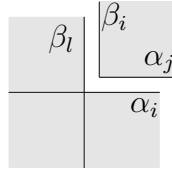


Figure 3: Locally unobstructed pair  $(\alpha_i, \alpha_j)$  and coerced pair  $(\beta_i, \beta_l)$

**Definition 3.25.** Let  $(\alpha_i, \alpha_j) \in F_i \times F_j \setminus \mathcal{L}$  be a matched pair which is globally unobstructed. We then define the welding along  $(\alpha_i, \alpha_j)$  by iteratively gluing as in Definition 3.21 along  $(\alpha_i, \alpha_j)$  and all its resulting coerced pairs.

**3.26** Using the welding method detailed above, we may construct any complete log affine manifold  $X$  with corners, with divisor  $D$  of normal crossing type, trivial affine monodromy, and such that each component of  $X \setminus D$  is identified with a standard affine space  $U$ .

### Examples

We now present three illustrative examples of welding to produce nontrivial log affine structures on  $\mathbb{R}^2$ ,  $S^2$ ,  $T^2$  and an orientable surface of genus two. In these examples we omit a detailed discussion of the resulting logarithmic one-form (8) and its residues or integrals.

**Example 3.27.** Consider a complete log affine manifold diffeomorphic to  $\mathbb{R}^2$ , with degeneracy along three curves with normal crossings as depicted in Figure 4.

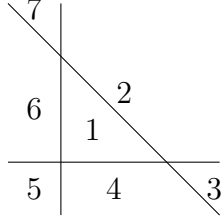


Figure 4: Log affine structure on  $\mathbb{R}^2$  with degeneracy along three skew lines

To build the example, we use seven partial compactifications of the affine plane, labeled as in Figure 4 and with fans as shown in Figure 5. The set of matched pairs depicted in Figure 5 give rise, after welding, to a complete log affine structure on a contractible 2-dimensional manifold without boundary, with degeneration locus as described above.

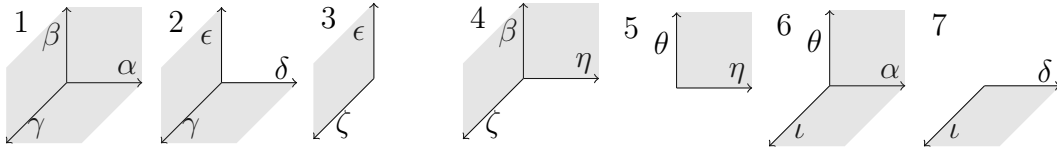


Figure 5: Welding of seven log affine manifolds with corners

**Example 3.28.** Expressing  $S^2$  as the quotient of  $\mathbb{R}^3 \setminus \{0\}$  by the diagonal  $\mathbb{R}$ -action

$$t \cdot (x, y, z) = e^t(x, y, z), \quad (23)$$

we see that  $S^2$  carries a residual  $\mathbb{R}^2$  action, and therefore a log affine structure with degeneracy locus given by the intersection with the coordinate hyperplanes, a normal crossing divisor which divides the sphere into 8 contractible regions.

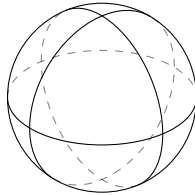


Figure 6: Log affine structure on  $S^2$  with degeneracy along three circles

We may therefore describe it as a welding of 8 copies of a triangular compactification of  $\mathbb{R}^2$ , with fans and matching pairs as described in Figure 7.

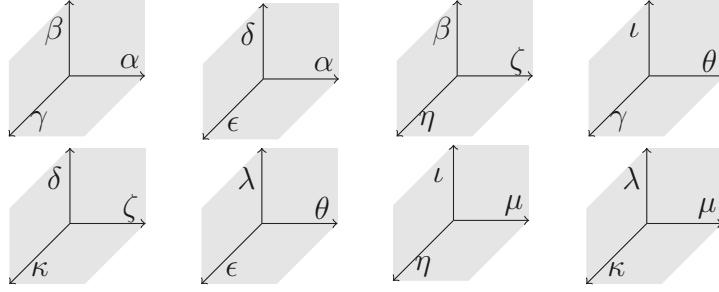


Figure 7: Log affine structure on  $S^2$  welded from eight triangles

**Example 3.29.** The 2-torus  $S^1 \times S^1$  carries a standard affine structure but we may also equip it with a log affine structure given by the action of  $\mathbb{R}^2$  generated by the vector fields  $V_1 = (X, 0)$  and  $V_2 = (0, X)$ , where  $X$  is a vector field on  $S^1$  with two nondegenerate zeros. The determinant  $V_1 \wedge V_2$  then defines a normal crossing divisor consisting of four circles, dividing the torus into four square regions. We may therefore describe this example by a welding of four squares, as in Figure 8.

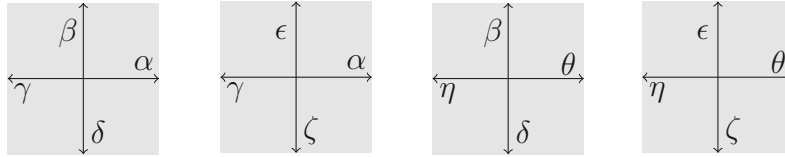


Figure 8: Log affine structure on  $T^2$  welded from four squares

**Example 3.30.** Fix a complete hexagonal fan in  $\mathbb{R}^2$  and weld four copies of this fan according to the matched pairs listed in Figure 9. The result is a complete log affine structure on the orientable genus 2 surface, corresponding to a tiling of the surface by four hexagons.

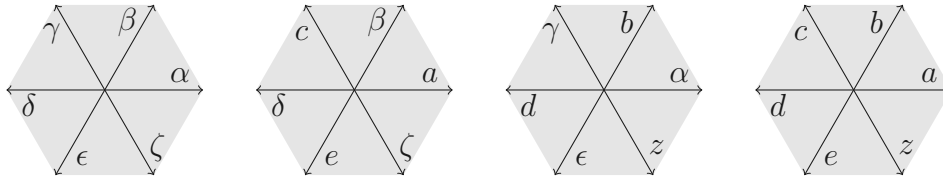


Figure 9: Log affine structure on genus 2 surface welded from four hexagons

## 4 Toric log symplectic manifolds

**4.1** In analogy with the Poisson structure on the dual of any Lie algebra, there is a canonical Poisson structure on the total space of the dual bundle of any Lie algebroid [4]. In the case of the tangent bundle, this coincides with the canonical symplectic form on the total space of the cotangent bundle. Given an action of  $\mathbb{R}^n$

on  $X$  generated by vector fields  $X_1, \dots, X_n$ , let  $A = \mathbb{R}^n \ltimes X$  be the corresponding abelian action Lie algebroid. If  $\{p_1, \dots, p_n\}$  are standard vertical coordinates on  $N = \text{tot}(A^*)$ , then the canonical Poisson structure is given by

$$\sum_{k=1}^n \frac{\partial}{\partial p_k} \wedge X_k \in C^\infty(N, \wedge^2 TN).$$

If  $X$  is taken to be a log affine manifold with free divisor  $D$ , then  $N \cong \mathbb{R}^n \times X$  is also log affine for the pullback divisor  $Z = \mathbb{R}^n \times D$ . The above Poisson structure is then nondegenerate as a section of  $\wedge^2(TN(-\log Z))$ . If  $\{\xi_1, \dots, \xi_n\}$  is the basis of log forms dual to the basis  $\{X_1, \dots, X_n\}$  of  $TX(-\log D)$ , then we may write the inverse of the above Poisson structure as follows:

$$\omega = \sum_{k=1}^n \xi_k \wedge dp_k \in \Omega^2(N, \log Z). \quad (24)$$

This generalization of a symplectic structure is the focus of our study, and we may characterize it as follows.

**Definition 4.2.** *Let  $M$  be a manifold with free divisor  $Z$  and corresponding Lie algebroid  $TM(-\log Z)$ . A logarithmic symplectic form is a closed logarithmic 2-form*

$$\omega \in \Omega^2(M, \log Z) \quad (25)$$

*which is nondegenerate in the sense that interior product defines an isomorphism*

$$\omega : TM(-\log Z) \rightarrow (TM(-\log Z))^*.$$

**Remark 4.3.** A log symplectic form, as defined above, may be viewed alternatively as a usual Poisson structure  $\pi$  which degenerates along  $Z$  in such a way that the top power  $\pi^n$  is a reduced defining section for the free divisor  $Z$ .

**4.4** The log symplectic form (24) is invariant under the global  $\mathbb{R}^n$  action on  $N$  by fibrewise translations. We may therefore quotient by the lattice  $\Gamma = (2\pi\mathbb{Z})^n \subset \mathbb{R}^n$  and obtain a (trivial) principal  $T^n$  bundle  $M$  equipped with a log symplectic form invariant under the torus, which we also denote by  $\Omega$ . The bundle projection then induces a smooth submersion

$$M \xrightarrow{\mu} X.$$

Let  $\mathfrak{t} \cong \mathbb{R}^n$  be the Lie algebra of  $T^n$  and let  $\rho : \mathfrak{t} \rightarrow C^\infty(M, TM)$  be the infinitesimal principal action, which takes values in the sections of  $TM(-\log \pi^*D)$ . By contraction, we have a natural closed logarithmic 1-form with values in  $\mathfrak{t}^*$ :

$$i_\rho \omega \in \Omega^1(M, \log Z) \otimes \mathfrak{t}^*.$$

In the case that  $Z$  is of normal crossing type, we may use the Mazzeo-Melrose decomposition A.4 to express the logarithmic cohomology as follows, summing over the smooth components of  $Z$ :

$$H^1(M, \log Z) \cong H^1(M) \oplus \sum_i H^0(Z_i). \quad (26)$$

Recall that as a log affine manifold,  $X$  carries a natural  $\mathfrak{t}^*$ -valued logarithmic form (c.f. Equation 2)  $\xi$ , and so the map  $\mu$  plays the role of a momentum map, in the sense that it satisfies

$$i_\rho \omega = -\mu^* \xi. \quad (27)$$

**Definition 4.5.** *A toric log symplectic manifold is a log symplectic manifold  $(M, Z, \omega)$  of dimension  $2n$ , equipped with an effective action of the torus  $T^n$  by symplectomorphisms, as well as a proper,  $T^n$ -invariant smooth map  $\mu : M \rightarrow X$  to a log affine manifold  $(X, D, \xi)$  called a tropical moment map, satisfying condition (27).*

*In the case that  $Z$  is a normal crossing divisor, we say that the toric log symplectic manifold is Hamiltonian when the logarithmic cohomology class of  $i_\rho \omega$  has vanishing component in  $H^1(M) \otimes \mathfrak{t}^*$ .*

**Remark 4.6.** The most significant aspect of the above definition is that while  $i_\rho \omega$  is closed, we do not require it to be *exact*, that is, to be the differential of a Hamiltonian function. The “reason” for this is that such Hamiltonians would have logarithmic singularities along  $Z$ , and there is no need to introduce functions with singularities in the development of the theory.

On the other hand, in the Hamiltonian case we do require that the component of  $[i_\rho \omega]$  in  $H^1(M) \otimes \mathfrak{t}^*$  vanishes, though the full logarithmic class may not. This condition specializes to the traditional Hamiltonian assumption in the case of usual symplectic manifolds.

**Remark 4.7.** One can show using Proposition A.19 that the vanishing of the component of  $[i_\rho \omega]$  in  $H^1(M) \otimes \mathfrak{t}^*$  is equivalent to the same condition on  $[\xi]$ , namely that its Mazzeo-Melrose component in  $H^1(X) \otimes \mathfrak{t}^*$  vanishes. This means that the Hamiltonian assumption above is equivalent to the assumption that  $(X, D, \xi)$  has trivial affine monodromy.

## 4.1 Trivial principal case

**4.8** In Paragraph 4.4 we described the simplest example of a toric log symplectic manifold: a trivial principal  $T^n$  bundle over a log affine manifold.

**Proposition 4.9.** *Let  $(X, D, \xi)$  be a log affine manifold, and let  $M$  be the trivial principal  $T^n$  bundle over  $X$ . The log symplectic form (24) then defines a toric log symplectic structure on  $M$  with momentum map  $\mu$  given by the bundle projection and degeneracy locus  $Z = \mu^{-1}(D)$ .*

**4.10** This example may be deformed by adding a *magnetic term*. That is, if  $\omega$  is the canonical symplectic form (24), then

$$\omega + \mu^* B$$

is also a log symplectic form, for any closed form  $B \in \Omega^2(X, \log D)$ . By an application of the Moser method, one can show that only the logarithmic cohomology class of  $B$  is relevant to the resulting symplectic structure up to equivariant symplectomorphism, yielding the following result.

**Theorem 4.11.** *Fix the log affine manifold  $(X, D, \xi)$  and the trivial principal  $T^n$  bundle  $\mu : M \rightarrow X$  as above, and let  $\omega_0$  be the canonical log symplectic form (24). If  $\omega$  is another log symplectic form such that  $(M, \omega, \mu)$  is toric log symplectic, then the difference  $\omega - \omega_0$  is basic and closed. Furthermore, the map*

$$\omega \mapsto [\omega - \omega_0] \in H^2(X, \log D)$$

*induces a bijection between equivalence classes of toric log symplectic structures with fixed momentum map  $\pi$  and the vector space  $H^2(X, \log D)$ .*

**Remark 4.12.** The usual notions of invariant and basic forms on a principal bundle carry over to the logarithmic context above, because of the exact sequence relating logarithmic vector fields on the domain and codomain of  $\pi$ :

$$0 \longrightarrow T_{M/X} \longrightarrow T_M(-\log \mu^* D) \xrightarrow{\mu^*} \mu^* T_X(-\log D) \longrightarrow 0, \quad (28)$$

where  $T_{M/X}$  is the vertical tangent bundle of the principal bundle  $M$ . This is an example of an algebroid submersion (see [7]) and gives rise to an injective cochain homomorphism from the log de Rham complex of  $(X, D)$  to the log de Rham complex of  $(M, \mu^* D)$ .

**Remark 4.13.** By an argument similar to that of Mazzeo-Melrose [16] (see Appendix A.4), one can show that for  $D \subset X$  a normal crossing divisor, the logarithmic cohomology groups may be expressed in terms of usual de Rham cohomology:

$$H^2(X, \log D) = H^2(X) \oplus \sum H^1(D_i) \oplus \sum H^0(D_i \cap D_j),$$

where we sum over the components  $D_i$  of  $D$  and their pairwise intersections.

For example, the log affine structure on  $S^2$  described in Example 3.28 has a divisor with three circular components intersecting in six points; as a result there is a 10-dimensional space of toric log symplectic structures on the trivial  $T^2$ -bundle.

## 4.2 Nontrivial principal case

**4.14** We now show that certain nontrivial principal  $T^n$  bundles over the log affine manifold  $(X, D, \xi)$  admit toric log symplectic structures.

**Definition 4.15.** *Let  $\pi : M \rightarrow X$  be a principal  $T^n$  bundle over the log affine  $n$ -manifold  $(X, D, \xi)$ , with real Chern classes  $c_1(M) = (c_1^1, \dots, c_1^n) \in H^2(X, \mathbb{R}) \otimes \mathfrak{t}$ . We define the toric log obstruction class of  $M$  to be the class*

$$\mathrm{Tr}(c_1(M) \wedge [\xi]) = \sum_{k=1}^n c_1^k \wedge [\xi_k] \in H^3(X, \log D). \quad (29)$$

**Theorem 4.16.** *With  $M$  and  $X$  as above, there exists a log symplectic form  $\omega$  making  $(M, \omega, \pi)$  a toric log symplectic manifold if and only if the toric log obstruction class (29) vanishes.*

*Under this condition, the space of equivalence classes of log symplectic forms making  $(M, \omega, \pi)$  toric log symplectic is an affine space modeled on the vector space  $H^2(X, \log D)$ .*



*Proof.* Let  $(M, \omega, \pi)$  be as above. Choosing a principal connection  $\theta \in \Omega^1(M) \otimes \mathfrak{t}^*$ , we may write

$$\omega = \sum_{k=1}^n \theta^k \wedge \alpha_k + \beta, \quad (30)$$

where  $\alpha_k, \beta$  are invariant and basic logarithmic forms. The momentum map condition (27) implies that  $\alpha_k = \xi_k$ . Taking derivatives, we obtain

$$\sum_{k=1}^n F^k \wedge \xi_k + d\beta = 0,$$

where  $[F^k] = c_1^k$ , so that (29) vanishes, as required.

Conversely, choose local trivializations for the principal  $T^n$  bundle  $M$  over an open cover  $\{U_i\}$  of  $X$ , and let

$$\varphi_{ij} : (U_i \times T^n)|_{U_i \cap U_j} \rightarrow (U_j \times T^n)|_{U_i \cap U_j}$$

be the transition isomorphisms satisfying the usual cocycle condition. On the  $k^{\text{th}}$  circle factor,  $\varphi_{ij}$  acts by multiplication by  $g_{ij}^k : U_i \cap U_j \rightarrow S^1$ . Also, choose a principal connection, defined by connection forms  $A_i^k \in \Omega^1(U_i, \log D)$  satisfying the cocycle condition  $i(A_i^k - A_j^k) = d \log g_{ij}^k$ .

We first equip each  $U_i \times T^n$  with the standard log symplectic form given in (24), namely

$$\tilde{\omega}_i = \sum_{k=1}^n \xi_k \wedge dp_k.$$

If we write  $g_{ij}^k = \exp(i\tau_{ij}^k)$  then we have

$$\varphi_{ij}^* \tilde{\omega}_j - \tilde{\omega}_i = \sum_{k=1}^n \xi_k \wedge d\tau_{ij}^k = \sum_{k=1}^n \xi_k \wedge (A_i^k - A_j^k). \quad (31)$$

If the class (29) vanishes, then there exists  $B \in \Omega^2(X, \log D)$  such that, over  $U_i$ ,

$$dB = \sum_{k=1}^n F^k \wedge \xi_k = \sum_{k=1}^n dA_i^k \wedge \xi_k. \quad (32)$$

Combining (31) and (32), we see that the modified symplectic forms

$$\omega_i = \tilde{\omega}_i + \mu^*(B|_{U_i} + \sum_{k=1}^n \xi_k \wedge A_i^k)$$

define a global form  $\omega$  on  $M$  rendering  $(M, \omega, \mu)$  toric Hamiltonian, as required.

Due to the choice of the 2-form  $B$ , we see that the symplectic form is only uniquely determined modulo closed logarithmic 2-forms on  $X$ . Applying the Moser method, which uses the properness of  $\mu$ , we therefore obtain a free and transitive action of  $H^2(X, \log D)$  on the set of equivalence classes of toric Hamiltonian log symplectic structures  $(M, \omega, \mu)$ , as needed.  $\square$

**Remark 4.17.** For  $D \subset X$  a normal crossing divisor, we have the following expression for the third logarithmic cohomology:

$$H^3(X, \log D) = H^3(X) \oplus \sum H^2(D_i) \oplus \sum H^1(D_i \cap D_j) \oplus \sum H^0(D_i \cap D_j \cap D_k),$$

where we sum over double and triple intersections of the components  $D_i$  of  $D$ . If we also have  $\dim X = 2$ , then this group vanishes.

**Example 4.18.** For the log affine structure on  $S^2$  described in Example 3.28, we have a 10-dimensional affine moduli space of toric log symplectic structures on any principal  $T^2$  bundle over  $S^2$ , including for example the Hopf manifold  $S^3 \times S^1$ .

**Example 4.19.** The log affine structure in Example 3.30 has a divisor with six components intersecting in six  $\mathbb{R}^2$  fixed points. As a result, we obtain a 13-dimensional affine moduli space of toric log symplectic structures on any principal  $T^2$  bundle over the orientable genus 2 surface.

**4.20** Theorem 4.16 provides a classification of toric log symplectic structures on a principal  $T^n$  bundle which satisfies condition (29). Note, however, that the notion of equivalence is that of equivariant symplectomorphism  $\psi : M \rightarrow M$ , in the sense that  $\mu \circ \psi = \mu$ . In some cases, there are additional symmetries which do not commute with  $\mu$ , and we describe these now.

The main observation is that while toric log symplectic forms on such a principal bundle may be deformed by closed basic forms, the map on cohomology groups has a kernel: we have the exact Gysin sequence

$$H^0(X, \log D) \otimes \mathfrak{t}^* \xrightarrow{c_1(M)} H^2(X, \log D) \xrightarrow{\mu^*} H^2(M, \log \pi^* D). \quad (33)$$

As a result, we may expect that 2-forms in the kernel of  $\mu^*$  would act trivially on the symplectomorphism class of  $(M, \omega)$ .

Indeed, let  $v = (v_1, \dots, v_n) \in \mathfrak{t}^*$  and consider the class

$$c_1(M) \cdot v = \sum_{k=1}^n v_k c_1^k \in H^2(X, \log D).$$

Choosing a connection  $\theta$  on  $M$  as in the proof of Theorem 4.16, the above class gives rise to a 1-parameter family of toric log symplectic structures

$$\omega_t = \omega + t \sum_{k=1}^n v_k F^k.$$

The Moser argument then indicates that the flow of  $X_t = \omega_t^{-1}(v \cdot \theta)$  trivializes the above family. This vector field is invariant and projects to the vector field  $\rho(v)$  on the log affine manifold  $X$ . Therefore,  $X_t$  is complete (allowing the Moser argument to proceed) if and only if  $\rho(v)$  is complete, yielding the following result.

**Theorem 4.21.** *If  $\omega, \omega'$  are two toric log symplectic structures on the principal  $T^n$ -bundle  $(M, \pi)$  over the complete log affine manifold  $(X, D, \xi)$ , and if  $[\omega' - \omega]$  is of the form  $c_1(M) \cdot v$  for  $v \in \mathfrak{t}$ , then there is a symplectomorphism  $\psi : (M, \omega) \rightarrow (M, \omega')$  such that  $\mu \circ \psi = T_v \circ \mu$ , where  $T_v : X \rightarrow X$  is the automorphism given by translation by  $v$ .*

## 5 Polytopes and symplectic cutting

In this section,  $(X, D, \xi)$  denotes a fixed complete log affine  $n$ -manifold, possibly with corners, and we assume that it is a tropical welded space of the type constructed in Section 3.2, so that all of its strata (i.e.  $\mathbb{R}^n$ -orbits) are affine spaces and  $D$  is of normal crossing type.

### 5.1 Log affine polytopes

**5.1** Each stratum of  $X$  is an affine space, where there is a clear notion of affine hyperplane. We extend this notion as follows.

**Definition 5.2.** *An affine linear hypersurface in the log affine manifold  $(X, D, \xi)$  is a connected and embedded hypersurface in  $X$ , not contained in  $D$ , and invariant under the action of a linear hyperplane in  $\mathfrak{t}$ .*

**Definition 5.3.** *A log affine polytope is a closed equidimensional submanifold with corners of  $X$  each of whose codimension 1 boundary strata has closure which is either:*

- *contained in  $D$ , and called a singular face, or*
- *contained in an affine linear hypersurface. Such a stratum may intersect  $D$ , in which case we require it to intersect all possible intersections of components of  $D$  transversely (so its union with  $D$  is also of normal crossing type), and we call the stratum a log face. If the stratum does not intersect  $D$ , we call it an interior face.*

*The polytope is called convex when its intersection with each component of  $X \setminus D$  is convex.*

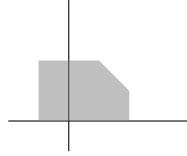


Figure 10: Convex log affine polytope with one singular, one interior, and three log faces

**5.4** Suppose the hypersurface  $H \subset X$  is not contained in  $D$ , but is affine linear for the hyperplane  $\mathfrak{t}_H \subset \mathfrak{t}$ . If  $H \cup D$  is normal crossing, then  $H$  inherits a log affine structure, with divisor  $D_H = D \cap H$  and trivialization  $\xi_H \in \Omega^1(H, \log D_H) \otimes \mathfrak{t}_H^*$  given by the projection to  $\mathfrak{t}_H^*$  of the pullback of  $\xi$  to  $H$ . As a result of this, we see that the closure of any log face contained in  $H$  is itself a log affine polytope.

Similarly, if  $C$  is a component of the degeneracy divisor  $D$ , then it inherits a log affine structure, with divisor  $D_C = (\overline{D \setminus C}) \cap C$  and trivialization  $\xi_C \in$

$\Omega^1(C, \log D_C) \otimes \mathfrak{t}_C^*$  given by the isomorphism of extensions

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & TX(-\log D)|_C & \longrightarrow & TC(-\log D_C) \\ \uparrow & & \uparrow \xi & & \uparrow \xi_C \\ \mathfrak{s}_C & \longrightarrow & \mathfrak{t} & \longrightarrow & \mathfrak{t}_C \end{array}$$

where  $\mathfrak{s}_C$  is the 1-dimensional stabilizer of a generic point on  $C$ . As a result, the closure of any singular face contained in  $C$  is also a log affine polytope.

**Proposition 5.5.** *The closure of any stratum of a log affine polytope is itself a log affine polytope.*

**5.6** Any interior face  $F$  of a log affine polytope is an affine polytope in the usual sense, and if it has dimension  $k$ , its associated 1-form  $\xi \in \Omega^1(F) \otimes \mathbb{R}^k$  defines a volume form  $\det \xi = \xi_1 \wedge \cdots \wedge \xi_k$ . As a result, any compact interior face has a well-defined volume in  $\mathbb{R}_+$ .

Even logarithmic volume forms may have well-defined integrals over compact manifolds, by extracting the Cauchy principal value: combining the results of [21] with Fubini's theorem (and using the normal crossing assumption on  $D$ ), we see that any compact and oriented log affine polytope has a well-defined volume, as long as none of its faces is contained in  $D$ .

**Definition 5.7.** *The regularized volume of a compact and oriented log affine polytope  $\Delta \subset (X, D, \xi)$  without singular faces is the real number defined by the iterated Cauchy principal value*

$$\text{Vol}(\Delta) = PV \int_{\Delta} \det \xi.$$

**Definition 5.8.** *An affine linear function on the log affine manifold  $(X, D, \xi)$  is a function  $f : X \rightarrow \mathbb{R}$  such that there exists a covector  $a \in (\mathbb{R}^n)^*$  (called the linear part of  $f$ ) such that for all  $u \in \mathbb{R}^n$ ,*

$$L_{\rho(u)} f = a(u), \tag{34}$$

*or, equivalently when  $X$  is complete,  $f(x + u) = f(x) + a(u)$  for all  $x \in X$  and translations  $u \in \mathbb{R}^n$ .*

**5.9** let  $\Delta \subset X$  be a log affine polytope, and let  $F \subset \Delta$  be any nonsingular face. Then there is an affine linear function  $f$  defined in a tubular neighbourhood  $U_F$  of  $F$  such that  $f|_F = 0$  and  $f \geq 0$  on  $\Delta \cap U_F$ . We call this a *boundary defining function* for the face  $F$ .

**Lemma 5.10.** *Let  $F$  be a log face with boundary defining function  $f$ . If  $F$  intersects a component  $D_i$  of the divisor  $D$  with associated residue  $v_i$ , then the linear part of  $f$  satisfies  $a(v_i) = 0$ .*

*Proof.* If  $p \in F \cap D_i$  then it is fixed by the action of  $v_i$ , and so  $L_{\rho(v_i)} f = a(v_i) = 0$ , as needed.  $\square$

**Definition 5.11.** An elementary log affine polytope is a convex log affine polytope  $\Delta$  with  $\Delta \setminus D$  connected.

**Lemma 5.12.** Let  $\Delta$  be an elementary log affine polytope, let  $F$  be a nonsingular face of  $\Delta$  and let  $f$  be a boundary defining function for  $F$  with linear part  $a$ . If a component  $D_i$  of the divisor  $D$  with associated residue  $v_i$  meets  $\Delta$  but does not intersect  $F$ , then  $a(v_i) < 0$ .

*Proof.* By convexity, each point  $p \in \Delta \setminus D$  lies on a unique straight line in the direction  $v_i$  whose closure meets  $D_i$  (at infinity in the  $-v_i$  direction). Since  $f$  is positive on the interior of  $\Delta$ , it follows that  $L_{\rho(v_i)}f < 0$ .  $\square$

**5.13** Let  $\Delta$  be an elementary log affine polytope in  $(X, D, \xi)$ , which we now assume is a tropical domain with associated simplicial fan  $\Sigma_F$  in  $U = \mathbb{R}^n$ , as in Definition 3.16. Uniquely defined up to a positive scalar, the boundary defining function  $f_i$  for each of the nonsingular faces  $F_i, i \in I$  of  $\Delta$  has linear part  $a_i$  which determines a half-space

$$H_i = \{u \in U : a_i(u) > 0\}.$$

By the above Lemmas,  $H_i$  does not intersect  $\Sigma_F$ . Compactness of  $\Delta$  may be expressed in terms of these half spaces, as follows:

**Proposition 5.14.** The elementary log affine polytope  $\Delta$  is compact if and only if

$$\Sigma_F \sqcup (\cup_{i \in I} H_i) = U.$$

**Example 5.15.** Let  $X$  be the tropical domain  $(\overline{\mathbb{R}^2})_A$ , for  $A = \{(1, 0), (1, 1)\}$ . Let  $\Delta \subset X$  be the log affine polytope given by the closure of  $\{f_1 \geq 0\} \cap \{f_2 \geq 0\}$ , for

$$f_1(x, y) = -y, \quad f_2(x, y) = -x + y.$$

Then  $\Delta$  is a compact convex log affine manifold with two singular faces and two log faces. Figure 11 shows the union of the fan defining  $X$  with the pair of half-planes defined by the nonsingular faces of  $\Delta \subset X$ .

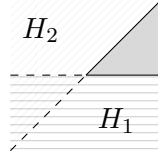


Figure 11: Fan and half-spaces defined by the compact polytope  $\Delta$

## 5.2 Symplectic cuts

**5.16** Let  $\Delta \subset X$  be a log affine polytope and let  $F \subset \Delta$  be a nonsingular face with boundary defining function  $f$ . Let  $(M, Z, \omega, \pi)$  be a principal toric Hamiltonian space over  $X$ , as in Theorem 4.16. If the linear part of  $f$  is a primitive integral

vector  $a \in \mathfrak{t} = \mathbb{R}^n$ , then the Hamiltonian function  $\pi^*f$  generates the action of a circle subgroup  $S_a^1 \subset T^n$  on  $M$  and we may define a log symplectic cut of  $M$  along  $\pi^{-1}(F)$ , generalizing the usual symplectic cut [15] as follows. Note that while  $f$  is only defined in a neighbourhood of  $F$ , this is irrelevant to the symplectic cut procedure, in which we only modify  $M$  near  $\pi^{-1}(F)$ . We assume for simplicity that  $\pi^*f$  is defined on all of  $M$ .

**Theorem 5.17.** *Let  $(M, Z, \omega)$  be a log symplectic manifold and let  $f : M \rightarrow \mathbb{R}$  be a Hamiltonian function generating an  $S^1$  action. Suppose that the  $S^1$  action is free along  $f^{-1}(0)$ .*

*Then  $M \times \mathbb{C}$ , equipped with the log symplectic form  $\omega + \text{id}z \wedge d\bar{z}$ , has an antidiagonal  $S^1$  action whose Hamiltonian  $\mu = f - |z|^2$  gives rise to a smooth log symplectic reduction*

$$M_f = \mu^{-1}(0)/S^1 \quad (35)$$

*called the log symplectic cut. The Hamiltonian  $f$  descends to a function  $\tilde{f}$  on  $M_f$ , defining a residual  $S^1$  action which now fixes the log symplectic submanifold  $S = \tilde{f}^{-1}(0)$ . As in the usual symplectic cut, we may identify  $S$  with the log symplectic reduction  $f^{-1}(0)/S^1$ , whereas  $M_f \setminus S$  is isomorphic to  $M \setminus f^{-1}(0)$ .*

*Proof.* Since the  $S^1$  action is free along  $f^{-1}(0)$ , the generating vector field  $\partial_\theta$  is nowhere vanishing there, and so by the nondegeneracy of  $\omega$ , along this locus  $df = -i(\partial_\theta)\omega$  is a nowhere vanishing log 1-form, showing that

$$df : TM(-\log Z)|_{f^{-1}(0)} \rightarrow T_0\mathbb{R}$$

is surjective, verifying the transversality condition of Proposition A.8. This immediately implies the transversality condition for  $\mu$  also holds. The same freeness assumption also implies that the antidiagonal  $S^1$  action on  $\mu^{-1}(0)$  is free, giving a smooth log symplectic quotient by Proposition A.8. The remaining statements follow from the same arguments given for usual symplectic cutting.  $\square$

**5.18** The above construction allows us, just as in the usual Delzant theory, to iteratively apply symplectic cuts along the nonsingular faces of  $\Delta$ , under the usual Delzant condition that whenever  $k$  such faces meet, we may choose boundary defining functions whose linear parts define an integral basis for a maximal rank  $k$  sublattice of  $\mathbb{Z}^n$ .

**Definition 5.19.** *The log affine polytope  $\Delta$  satisfies the Delzant condition when, for any point  $p \in \Delta$ , the collection of  $k$  nonsingular faces meeting  $p$  defines an integral basis for a maximal rank  $k$  sublattice of  $\mathbb{Z}^n \subset \mathfrak{t}$ .*

**Corollary 5.20.** *Given any log affine polytope  $\Delta$  in the log affine manifold  $(X, D, \xi)$ , and given Chern classes  $(c_1^1, \dots, c_1^n) \in H^2(X, \mathbb{R}) \otimes \mathfrak{t}$  such that (29) vanishes, we may construct a corresponding principal toric Hamiltonian log symplectic manifold  $(\widetilde{M}, \widetilde{Z}, \widetilde{\omega})$  with  $Z$  of normal crossing type and momentum map to  $X$  as in Theorem 4.16.*

If  $\Delta$  satisfies the Delzant condition, we may apply symplectic cuts to  $\widetilde{M}$  along all of the nonsingular faces of  $\Delta$  to produce another toric Hamiltonian log symplectic manifold  $(M, Z, \omega)$ , also with  $Z$  of normal crossing type, with an identical momentum map image. The toric orbit type stratification of  $(M, Z)$  then coincides with the stratification of  $\Delta$ , via the moment map.

If, in addition,  $\Delta$  has no singular faces, then  $(M, Z, \omega)$  is a smooth log symplectic manifold without boundary.

**Example 5.21.** On the log affine manifold  $(X, D, \xi)$  constructed in Example 3.30, we choose an affine linear function  $h$  whose linear part vanishes on the vectors  $\alpha = a = -\delta = -d$  and is positive on  $\epsilon = e$  and  $\zeta = f$ . The function  $h$  defines a half-space in each of the four tropical domains of  $X$ , defining a compact log affine polytope  $\Delta$  with a single log face and no singular faces. In Figure 5.21, we show the fans corresponding to the strata of  $X$  which meet  $\Delta$ , as well as the half-spaces  $H_1, \dots, H_4$  defined by the linear parts of  $h$  in each tropical domain. In Figure 5.21, we show the orientable genus 2 surface  $X$  together with its decomposition into four hexagonal domains, and we see that the log affine polytope  $\Delta$  is 2-dimensional submanifold of genus 1 with a single log face.

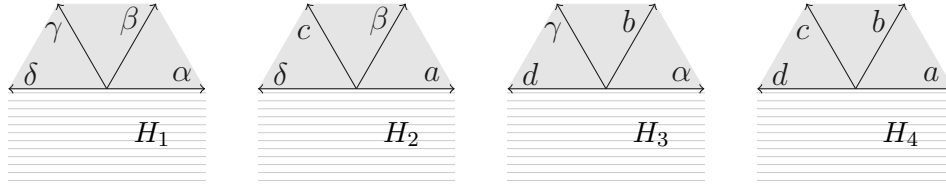


Figure 12: Fan and half-spaces defined by polytope  $\Delta$  in Example 5.21

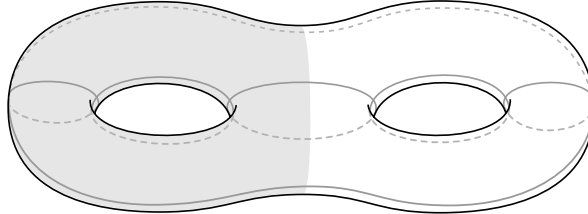


Figure 13: Log affine polytope of genus 1 with single log face, sitting in log affine manifold of genus 2 obtained from welding four hexagons.

This example demonstrates that log affine polytopes may have nontrivial topology and so toric Hamiltonian log symplectic manifolds need not be equivariantly diffeomorphic to usual toric symplectic manifolds, or even quasitoric manifolds. In fact, if we take the trivial principal  $T^2$  bundle over the polytope  $\Delta$  above and perform a symplectic cut along the unique log face, we conclude by Proposition 1 in [11] that the resulting manifold is diffeomorphic to  $S^1 \times ((S^1 \times S^2) \# (S^1 \times S^2))$ , and by Lemma 2.3 of [17], this manifold, although

it is almost complex, does not admit any symplectic structure, by an argument involving the vanishing of Seiberg-Witten invariants. Nevertheless, it admits a log symplectic structure by our construction.

## 6 Delzant correspondence

**6.1** In this section, we establish an analogue of the Delzant correspondence for a class of toric Hamiltonian log symplectic manifolds. The correspondence differs from Delzant's in several important ways: first, the analog of the Delzant polytope is a log affine polytope  $\Delta$  sitting in a tropical welded space, and need not be contractible; second, the  $n$ -dimensional polytope  $\Delta$  is decorated by an  $n$ -tuple of integral classes in  $H^2(\Delta)$ ; finally, with the above data fixed, the equivariant isomorphism class of the log symplectic form is unique only up to addition of a magnetic term coming from the logarithmic cohomology of  $\Delta$ .

**6.2** We prove an analogue of the Delzant classification for compact orientable log symplectic manifolds  $(M, Z, \omega)$ , under the assumption that  $Z$  is a real divisor of normal crossing type and such that  $\partial M \subset Z$ . We assume that there is an effective action of  $T^n$  preserving the structure and which is *Lagrangian*, meaning that the infinitesimal action  $\rho : \mathfrak{t} \rightarrow C^\infty(M, TM)$  satisfies  $\omega(\rho(a), \rho(b)) = 0$  for all  $a, b \in \mathfrak{t}$ .

**Theorem 6.3.** *Let  $(M, Z, \omega)$  be an orientable compact and connected log symplectic manifold with  $Z$  of normal crossing type and  $\partial M \subset Z$ , equipped with an effective Lagrangian  $T^n$  action. Then the following hold:*

1. *The quotient  $M/T^n$  defines a compact and connected Delzant log affine polytope  $(\Delta, D, \xi)$  (with  $D$  of normal crossing type). Furthermore, the quotient map  $\mu : M \rightarrow \Delta$  is a tropical momentum map.*
2. *The equivariant isomorphism class of  $(M, Z, \omega, \mu)$  is determined, up to translation by  $H^2(\Delta, \log D)$ , by the equivalence class of the log affine polytope  $\Delta$  and the  $n$ -tuple of Chern classes of  $\tilde{M}$ , the symplectic uncut of  $M$ .*

*If, in addition, the action is Hamiltonian in the sense of Definition 4.5, so that  $(\Delta, D, \xi)$  has trivial affine monodromy, then we also have:*

3.  *$\Delta$  is equivalent to a convex log affine polytope in a tropical welded space as in Section 3.2.*

*In particular,  $(M, Z, \omega)$  is equivariantly isomorphic to a toric Hamiltonian log symplectic manifold of the type constructed in Corollary 5.20.*

*Proof.*

*Part 1:* The notion of symplectic uncut was introduced in [18] to give a simpler proof of the Delzant correspondence for toric symplectic manifolds with proper momentum maps. It was studied carefully in [13] in order to characterize non-compact toric symplectic manifolds. We modify the construction slightly in Appendix A.3 in order to apply it to log symplectic manifolds.



The symplectic uncut  $(\widetilde{M}, \widetilde{Z}, \widetilde{\omega})$ , as described in Definition A.20, is a modification of  $(M, Z, \omega)$  along its submaximal orbit strata which renders the Lagrangian  $T^n$  action into a principal action. It follows from Lemma A.14 that  $\Delta$  is a compact manifold with corners and that  $D = Z/T^n$  is a normal crossing divisor.

Since the infinitesimal action  $\rho$  is Lagrangian, we have that the logarithmic 1-form

$$\iota_\rho \widetilde{\omega} \in \Omega^1(\widetilde{M}, \log \widetilde{Z}) \otimes \mathfrak{t}^*$$

is basic, defining a closed logarithmic 1-form  $\xi \in \Omega^1(\Delta, \log D) \otimes \mathfrak{t}^*$  which endows  $\Delta$  with the structure of a log affine manifold.

To see that  $\Delta$  is a log affine polytope, note that each component of  $\partial\Delta$  not contained in  $D$  is the  $T^n$  quotient of an exceptional boundary component of  $\widetilde{M}$ . By the normal form given by Lemma A.16, and by the computation in Example A.18, we see that each exceptional boundary component is the zero locus of an affine linear function with linear part given by a primitive generator of the integral lattice in  $\mathfrak{t}$ . Finally, Proposition A.17 implies that  $\Delta$  satisfies the Delzant condition required by Definition 5.19.

*Part 2:* The uncut construction is such that  $(M, Z, \omega)$  may be obtained from  $(\widetilde{M}, \widetilde{Z}, \widetilde{\omega})$  by symplectic cutting along the nonsingular faces of  $\Delta$ . By Theorem 4.16, the equivariant isomorphism class of the uncut is determined, up to translation by  $H^2(\Delta, \log D)$ , by the equivalence class of  $\Delta$  and the  $n$ -tuple of Chern classes of  $\widetilde{M}$ . Since equivariant isomorphisms descend under symplectic cutting, we obtain the result.

*Part 3:* We first construct a tropical welded space  $(X, D_X, \xi_X)$  which will contain the polytope  $(\Delta, D, \xi)$ . We construct  $X$  by welding a collection of tropical domains  $\{\mathfrak{t}_j^*\}$  corresponding to the connected components  $\{M_j\}$  of  $M \setminus Z$ . Each tropical domain  $\mathfrak{t}_j^*$  is constructed as in Definition 3.16, from a set  $F_j$  of vectors and a fan  $\Sigma_j$  defined as follows.

Each component  $Z_i \subset Z$  of the normal crossing divisor has an induced normal crossing divisor  $Z|_i = (\overline{Z \setminus Z_i}) \cap Z_i$ , and the residue of  $\omega$  defines a form  $\text{Res}_{Z_i}(\omega) \in \Omega^1(Z_i, \log Z|_i)$  which is  $T^n$ -invariant, so that

$$di_\rho \text{Res}_{Z_i}(\omega) = 0.$$

This implies that  $\iota_\rho \text{Res}_{Z_i}(\omega) \in \Omega^0(Z_i, \log Z|_i) \otimes \mathfrak{t}^*$  is a constant:

$$v_i = i_\rho \text{Res}_{Z_i}(\omega).$$

Each of the vectors  $v_i$  is nonzero, since  $T^n$  acts effectively on  $Z_i$ , as shown in the proof of Lemma A.14. The set  $F_j$  of vectors used for the compactification  $\mathfrak{t}_j^*$  is then the set of all  $v_i$  such that  $\overline{M_j} \cap Z_i \neq \emptyset$ . The associated fan  $\Sigma_j$  in  $\mathfrak{t}^*$  consists of all cones generated by subsets  $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$  of  $F_j$  with the property that

$$\overline{M_j} \cap Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_k} \neq \emptyset.$$

Such subsets are linearly independent by Proposition A.5.

Each component of  $Z$  which separates  $M_j$  from  $M_{j'}$  gives rise to a matched pair in  $F_j \times F_{j'}$ , and in this way we obtain a list of globally unobstructed matched pairs which defines a tropical welded space  $(X, D_X, \xi_X)$  as in Definition 3.25.

We now construct an embedding  $\iota$  of  $\Delta$  into  $X$  as a log affine submanifold; the embedding is unique up to an overall translation. Choose a basepoint  $x_0$  in a component  $\Delta_0$  of  $\Delta \setminus D$ , and choose  $y_0$  in the interior of the corresponding tropical domain  $\bar{t}_0^* \subset X$ . We define  $\iota(x_0) = y_0$ , which fixes the translation ambiguity. For any other point  $x \in \Delta \setminus D$ , we choose a path  $\gamma$  from  $x_0$  to  $x$  which, for convenience, is transverse to each component of  $D$ .

We may cover  $\gamma$  by open sets in each of which we have coordinates such that  $\xi$  is in normal form (46), and by applying Theorem A.6 we obtain a unique extension of  $\iota$  to the union  $U_\gamma$  of this cover of  $\gamma$ ,

$$\iota_\gamma : U_\gamma \rightarrow X, \quad (36)$$

which is a local diffeomorphism onto its image and preserves log affine structure. We write  $y = \iota_\gamma(x) \in X$ .

We now show that the construction above is path-independent. Let  $\gamma'$  be another such path from  $x_0$  to  $x$ , yielding a possibly different point  $y' = \iota_{\gamma'}(x) \in X$ . Then  $\iota_{\gamma'}(\gamma') \cdot \iota_\gamma(\gamma^{-1})$  is a path from  $y$  to  $y'$ . Because  $\Delta$  has trivial affine monodromy, the loop  $\gamma' \cdot \gamma^{-1}$  has zero regularized length, implying

$$\int_{\iota_{\gamma'}(\gamma') \cdot \iota_\gamma(\gamma^{-1})} \xi_X = \int_{\gamma' \cdot \gamma^{-1}} \xi = 0. \quad (37)$$

But by the construction of  $X$ ,  $y$  and  $y'$  lie in the same affine stratum  $A$  of  $X$ , so we may choose a path  $\delta \subset A$  from  $y$  to  $y'$ , and then using the affine structure on  $A$ , we have

$$y' - y = \int_\delta \xi_A = \int_{\iota_{\gamma'}(\gamma') \cdot \iota_\gamma(\gamma^{-1})} \xi_X = 0,$$

where  $\xi_A$  is the 1-form defining the affine structure on  $A$ , obtained from  $\xi$  by reduction and pullback as in Paragraph 5.4. In this way, we obtain a local diffeomorphism of log affine manifolds onto its image

$$\iota : \Delta \rightarrow X. \quad (38)$$

Because  $\Delta$  is compact, (38) is a covering map, so it suffices to show that it is a 1-sheeted cover. Take  $y_0 \in \iota(\Delta) \cap (X \setminus D_X)$  and  $x_0, x'_0 \in \iota^{-1}(y_0)$ . Because  $x_0$  and  $x'_0$  are in the same component of  $\Delta \setminus D$ , we may choose a path  $\lambda$  from  $x_0$  to  $x'_0$  contained in that component. Since  $\Delta$  has trivial monodromy, it follows that

$$x'_0 - x_0 = \int_\lambda \xi = 0,$$

as required.  $\square$

## A Appendix

### A.1 Local normal forms for log affine manifolds

**A.1** Let  $(X, D, \xi)$  be a log affine manifold with  $D$  of normal crossing type. If  $x \in X$  is such that precisely  $k$  components  $D_1, \dots, D_k$  of  $D$  meet  $x$ , then we may choose coordinates  $(y_1, \dots, y_n)$  in a neighbourhood  $U$  centered at  $x$  such that  $D$  is the vanishing locus of the monomial  $y_1 \cdots y_k$ . The coordinates also provide a natural trivialization of the algebroid  $TX(-\log D)$ , given by

$$TX(-\log D) = \langle y_1 \partial_{y_1}, \dots, y_k \partial_{y_k}, \partial_{y_{k+1}}, \dots, \partial_{y_n} \rangle. \quad (39)$$

**Proposition A.2.** *There exist coordinates  $(x_1, \dots, x_n)$  such that  $D$  is the vanishing locus of  $x_1 \cdots x_k$  and*

$$\xi = \sum_{i=1}^k x_i^{-1} dx_i \otimes v_i + \sum_{i=k+1}^n dx_i \otimes v_i, \quad (40)$$

for  $(v_1, \dots, v_n)$  a constant basis of  $\mathbb{R}^n$ .

*Proof.* Begin with coordinates  $(y_1, \dots, y_n)$  in the neighbourhood  $U$  as above, so that  $D$  is given by  $y_1 \cdots y_k = 0$  and

$$\xi = \sum_{i=1}^k y_i^{-1} dy_i \otimes \alpha_i + \sum_{i=k+1}^n dy_i \otimes \alpha_i,$$

where  $\alpha_i : U \rightarrow \mathbb{R}^n$  are smooth vector-valued functions. It follows from  $d\xi = 0$  that for each  $i \leq k$ ,  $\alpha_i$  is constant along the hypersurface  $y_i = 0$ . Denoting the basis  $(\alpha_1(0), \dots, \alpha_n(0))$  by  $(v_1, \dots, v_n)$ , we conclude that

$$\xi - \left( \sum_{i=1}^k y_i^{-1} dy_i \otimes v_i + \sum_{i=k+1}^n dy_i \otimes v_i \right) = dF,$$

for  $F : U \rightarrow \mathbb{R}$  smooth and  $F(0) = dF(0) = 0$ .

We now expand  $F$  in terms of the basis  $(v_1, \dots, v_n)$ :

$$F = \sum_{i=1}^n f_i v_i,$$

and we define new coordinates on a possibly smaller neighbourhood:

$$x_i = \begin{cases} y_i e^{f_i} & i \leq k \\ y_i + f_i & i > k. \end{cases} \quad (41)$$

In these coordinates we have the required expression (40).  $\square$

**A.3** In the coordinates provided by Proposition A.2, the  $\mathbb{R}^n$ -action on  $X$  may be written as

$$u \cdot (x_1, \dots, x_n) = (e^{v_1^*(u)} x_1, \dots, e^{v_k^*(u)} x_k, x_{k+1} + v_{k+1}^*(u), \dots, x_n + v_n^*(u)), \quad (42)$$

where  $(v_1^*, \dots, v_n^*)$  is the dual basis to  $(v_1, \dots, v_n)$ .

**A.4** We see from (40) that for each  $i \leq k$ ,  $v_i$  is the residue of  $\xi$  along  $D_i$ , and so is an invariant of the log affine manifold. We also see that these residues are linearly independent, as they form part of the basis  $(v_1, \dots, v_n)$ . The remaining basis elements  $\{v_i : i > k\}$ , however, are not invariant and may be adjusted by a change of coordinates, as follows.

If  $(v_1, \dots, v_k, w_{k+1}, \dots, w_n)$  is any other basis, define  $\{r_{ij}, s_{ij}\}$  via

$$v_j = \sum_{i>k} r_{ij} w_i + \sum_{i \leq k} s_{ij} v_i, \text{ for all } j > k. \quad (43)$$

We then make the coordinate change

$$\tilde{x}_i = \begin{cases} x_i \exp(\sum_{j>k} s_{ij} x_j) & \text{for } i \leq k \\ \sum_{j>k} r_{ij} x_j & \text{for } i > k. \end{cases} \quad (44)$$

As a consequence, we obtain

$$\sum_{i=1}^k x_i^{-1} dx_i \otimes v_i + \sum_{i=k+1}^n dx_i \otimes v_i = \sum_{i=1}^k \tilde{x}_i^{-1} d\tilde{x}_i \otimes v_i + \sum_{i=k+1}^n d\tilde{x}_i \otimes w_i. \quad (45)$$

**Proposition A.5.** *Let  $(X, D, \xi)$  be a log affine  $n$ -manifold and let  $x \in X$  meet exactly  $k$  components  $D_1, \dots, D_k$  of the normal crossing divisor  $D$ . The residues  $v_i \in \mathbb{R}^n$  of  $\xi$  along  $D_i$  then form a linearly independent set, and for any extension  $(v_1, \dots, v_k, v_{k+1}, \dots, v_n)$  to a basis, there are coordinates near  $x$  such that*

$$\xi = \sum_{i=1}^k x_i^{-1} dx_i \otimes v_i + \sum_{i=k+1}^n dx_i \otimes v_i. \quad (46)$$

**Theorem A.6.** *Let  $(X, D, \xi)$  and  $p \in U \subset X$  be as above. If we choose a basepoint  $q \in U \setminus D$  and map it to any point  $\phi(q) \in \mathbb{R}^n$ , then there is a unique extension of  $\phi$  to a map to the log affine manifold constructed in (18):*

$$\phi : U \rightarrow (\widetilde{\mathbb{R}^n})_A, \quad (47)$$

where  $A = \{v_1, \dots, v_k\}$  is the set of residues of  $\xi$  in  $U$ . This map is an isomorphism of log affine manifolds onto its image.

*Proof.* Choose local coordinates  $(x_1, \dots, x_n)$  in  $U$  so that  $\xi$  is given by (40), and choose similar adapted coordinates  $(y_1, \dots, y_n)$  for  $(\widetilde{\mathbb{R}^n})_A$ , so that its log 1-form  $\xi_A$  has the form

$$\xi_A = \sum_{i=1}^k y_i^{-1} dy_i \otimes v_i + \sum_{i=k+1}^n dy_i \otimes w_i, \quad (48)$$

for a basis  $(v_1, \dots, v_k, w_{k+1}, \dots, w_n)$  of  $\mathbb{R}^n$ . By Proposition A.5, we may change coordinates so that

$$\xi_A = \sum_{i=1}^k \tilde{y}_i^{-1} d\tilde{y}_i \otimes v_i + \sum_{i=k+1}^n d\tilde{y}_i \otimes v_i. \quad (49)$$

As a result, the map  $\phi$ , which must satisfy  $\phi^* \xi_A = \xi$ , is given by

$$\tilde{y}_i = \begin{cases} e^{c_i} x_i & \text{for } i \leq k \\ x_i + c_i & \text{for } i > k, \end{cases} \quad (50)$$

where the constants  $\{c_i \in \mathbb{R}\}$  are fixed by  $\phi(p) = q$ , as required.  $\square$

## A.2 Log symplectic reduction

**A.7** Let  $(M, Z, \omega)$  be a log symplectic manifold with an  $S^1$  action generated by the vector field  $\partial_\theta$ , and suppose it is Hamiltonian, in the sense that there is a function  $\mu : M \rightarrow \mathbb{R}$  such that

$$i_{\partial_\theta} \omega = -d\mu. \quad (51)$$

We impose the following transversality condition: along  $\widetilde{M} = \mu^{-1}(0)$ , the composition  $\widetilde{d\mu}$  of  $d\mu$  with the anchor map  $TM(-\log Z) \rightarrow TM$

$$TM(-\log Z)|_{\widetilde{M}} \xrightarrow{\widetilde{d\mu}} \mu^* T_0 \mathbb{R} \quad (52)$$

must be surjective. In particular, this implies that 0 is a regular value of  $\mu$  and so  $\widetilde{M} \subset M$  is a smooth hypersurface. If  $Z$  is a normal crossing divisor, (52) is surjective if and only if 0 is a regular value, not only for  $\mu$ , but also for the restriction of  $\mu$  to any stratum of  $Z$ . As a result, the intersection  $\widetilde{Z} = Z \cap \widetilde{M}$  is a normal crossing divisor in  $\widetilde{M}$ , and the kernel of the morphism (52) is the Lie algebroid  $T\widetilde{M}(-\log \widetilde{Z})$ .

If we assume further that  $S^1$  acts freely and properly on  $\widetilde{M}$ , we obtain a normal crossing divisor  $Z_0 = \widetilde{Z}/S^1$  in the quotient manifold  $M_0 = \widetilde{M}/S^1$ , and the quotient map  $\pi : \widetilde{M} \rightarrow M_0$  induces a Lie algebroid morphism and an exact sequence over  $\widetilde{M}$

$$0 \longrightarrow \langle \partial_\theta \rangle \longrightarrow T\widetilde{M}(-\log \widetilde{Z}) \xrightarrow{\pi_*} TM_0(-\log Z_0) \longrightarrow 0 \quad (53)$$

As in the usual Marsden-Weinstein symplectic reduction, the corank 1 subalgebroid  $T\widetilde{M}(-\log \widetilde{Z}) \subset TM(-\log Z)$  is coisotropic with respect to the log symplectic form, whose restriction has rank 1 kernel generated by  $\partial_\theta$ . Therefore, the pullback of  $\omega$  to  $\widetilde{M}$  is basic relative to  $\pi$ , and may be expressed as the pullback of a unique log symplectic form  $\omega_0 \in \Omega^2(M_0, \log Z_0)$ , defining the logarithmic symplectic reduction. We state a slight extension of this result for arbitrary free divisors.

**Proposition A.8.** *Let  $(M, Z, \omega)$  be a log symplectic manifold, let  $\mu$  be a Hamiltonian generating a circle action via (51), and assume that (52) is surjective, which implies that  $\mu^{-1}(0)$  is smooth, with free divisor  $\tilde{Z} = \mu^{-1}(0) \cap Z$ . Assume the  $S^1$  action on  $\mu^{-1}(0)$  is free and proper, with quotient  $M_0$  containing the free divisor  $Z_0 = \tilde{Z}/S^1$ . Then the pullback of  $\omega$  to  $\mu^{-1}(0)$  is basic in the sense of (53), and defines a logarithmic symplectic structure  $(M_0, Z_0, \omega_0)$ , called the log symplectic quotient.*

### A.3 Symplectic uncut

Let  $(M, Z, \omega)$  be a compact orientable log symplectic manifold with corners and with  $Z$  of normal crossing type, equipped with an effective  $T^n$  action. We construct a log symplectic principal  $T^n$  bundle  $(\tilde{M}, \tilde{Z}, \tilde{\omega})$  over the quotient  $M/T^n$ , called the symplectic uncut (Definition A.20), by applying the compressed blow-up operation to  $M$  along its submaximal orbit type strata, following an idea of Meinrenken [18]. This operation is inverse to the symplectic cut introduced in Section 5.2, which may be applied to  $\tilde{M}$  along  $\tilde{Z}$  to recover  $M$ .

#### Compressed blow-up

To define the compressed blow-up operation (Definition A.11), we first recall the blow-up operation and the boundary compression construction.

**A.9** For  $M$  a manifold and  $K$  a submanifold of codimension 2, we denote the *real oriented blow-up* of  $M$  along  $K$  by  $\text{Bl}_K(M)$ . We denote the exceptional divisor by  $E$ , which is a boundary component of  $\text{Bl}_K(M)$ . The blow-down map

$$p : \text{Bl}_K(M) \rightarrow M \tag{54}$$

restricts to a diffeomorphism on  $\text{Bl}_K(M) \setminus E$  and expresses  $E$  as an  $S^1$  fibre bundle over  $K$ .

**A.10** We now define a *compression* operation closely related to the unfolding defined in [3]. Let  $M$  be a smooth manifold with corners and let  $Z$  be a smooth component of the boundary. Let  $U \cong Z \times [0, \epsilon)$  be a tubular neighbourhood of  $Z$ , and let  $V = M \setminus Z$ . Then  $M$  is the fibered coproduct, or gluing, of  $U$  and  $V$  along  $U \cap V$ . Consider the open embedding

$$\iota : U \cap V \cong Z \times (0, \epsilon) \rightarrow \tilde{U} = Z \times [0, \tfrac{1}{2}\epsilon^2), \quad (p, r) \mapsto \left(p, \tfrac{1}{2}r^2\right). \tag{55}$$

The fibered coproduct of  $\tilde{U}$  and  $V$  along  $U \cap V$  is a smooth manifold with corners  $\tilde{M}$ . The identity map on  $V$  extends to a smooth homeomorphism  $q : M \rightarrow \tilde{M}$  such that the restriction to  $Z$  is a diffeomorphism onto its image  $\tilde{Z} = q(Z)$ . The triple  $(\tilde{M}, \tilde{Z}, q)$  is called the *compression* of  $(M, Z)$ .

We now compress the blow-up along its exceptional divisor to obtain the following.

**Definition A.11.** Let  $M$  be a manifold with corners and let  $K \subset M$  be a submanifold of codimension 2. Then the compressed blow-up of  $M$  along  $K$  is the compression  $(\widetilde{M}_K, \widetilde{E}, q)$  of  $(\text{Bl}_K(M), E)$ .

If  $S \subset M$  is a submanifold that intersects  $K$  cleanly, the compressed proper transform of  $S$  is the submanifold

$$\widetilde{S} = \overline{q \circ p^{-1}(S \setminus K)} \subset \widetilde{M}_K.$$

**A.12** We may now apply the compressed blow-up operation to the submaximal strata of an effective  $T^n$  action. Let  $M$  be a  $2n$ -dimensional compact orientable manifold with corners equipped with an effective  $T^n$  action. By [9, Corollary B.48], the fixed point set of a circle subgroup  $H \subset T^n$  is a disjoint union of closed submanifolds of codimension at least 2.

Let  $H_1, H_2, \dots, H_m \subset T^n$  be the circle subgroups such that the fixed point set  $M^{H_i}$  contains at least one codimension 2 component (there are finitely many  $H_i$ , because  $M$  is compact, and therefore has finitely many orbit type strata). Let  $K_i$  be the union of the codimension 2 components of  $M^{H_i}$ .

Let  $\widetilde{M}_{K_1}$  be the compressed blow-up of  $M$  along  $K_1$ . The compressed proper transform

$$\widetilde{K}_2 \subset \widetilde{M}_{K_1}$$

is a codimension 2 submanifold of  $\widetilde{M}_{K_1}$ , so we may apply the compressed blow-up operation to  $\widetilde{M}_{K_1}$  along  $\widetilde{K}_2$ . Continuing in this way, we obtain the following.

**Definition A.13.** Let  $M$  be a  $2n$ -dimensional compact orientable manifold with corners with an effective  $T^n$  action. The iterated compressed blow-up  $\widetilde{M}$  is the manifold with corners obtained from  $M$  by successively applying the compressed blow-up operation to the fixed point sets  $\{K_1, K_2, \dots, K_m\}$ .

The iterated compressed blow-up  $\widetilde{M}$  receives the compression map

$$q : \text{Bl}_K(M) \rightarrow \widetilde{M}$$

where  $\text{Bl}_K(M)$  is the iterated real oriented blow-up of  $M$  along  $\{K_1, K_2, \dots, K_m\}$ .

**Lemma A.14.** Let  $M$  be a  $2n$ -dimensional compact orientable manifold with corners endowed with an effective  $T^n$  action. The iterated compressed blow-up  $\widetilde{M}$  in Definition A.13 is a principal  $T^n$  bundle over the quotient  $\Delta = M/T^n$ , and therefore  $\Delta$  is a compact orientable manifold with corners. Furthermore, the following diagram commutes

$$\begin{array}{ccc} \text{Bl}_K(M) & \xrightarrow{p} & M \\ q \downarrow & & \downarrow \\ \widetilde{M} & \longrightarrow & \Delta \end{array} \quad (56)$$

and the blow-down and folding maps  $p, q$  are  $T^n$ -equivariant.

In addition, if  $Z$  is a  $T^n$ -invariant normal crossing divisor in  $M$ , then the compressed proper transform  $\widetilde{Z}$  is a normal crossing divisor in  $\widetilde{M}$ , and  $D = Z/T^n$  is a normal crossing divisor in  $\Delta$ .

*Proof.* Because  $M$  is orientable,  $T^n$  acts freely on the principal orbit stratum, and every point with non-trivial stabilizer is contained in some  $K_i$ .

For any point  $x \in K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_k}$ , there exists a  $T^n$ -invariant neighbourhood  $U_x$  that is equivariantly isomorphic to a neighbourhood of

$$(x, 0) \in (U_x \cap K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_k}) \times \mathbb{C}^k,$$

where  $\mathbb{C}^k$  is endowed with the standard  $U(1)^k$  action. The preimage  $q \circ p^{-1}(x)$  is equivariantly isomorphic to a  $k$ -torus  $T^k$ , and  $q \circ p^{-1}(U_x)$  is equivariantly isomorphic to a neighbourhood of

$$q \circ p^{-1}(x) \cong \{(x, 0)\} \times T^k \in (U_x \cap K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_k}) \times \mathbb{R}_+^k \times T^k,$$

where  $\mathbb{R}_+ = [0, \infty)$ . Therefore, the induced  $T^n$  action on  $\widetilde{M}$  is free and proper, and diagram (56) commutes.

Since  $M$  is orientable, it follows that the iterated compressed blow-up  $\widetilde{M}$  is orientable, and therefore  $\Delta$  is orientable.

Let  $Z_1, Z_2, \dots, Z_l$  be components of  $Z$ . The intersection

$$L = \bigcap_{i=1, \dots, l} Z_i$$

is  $T^n$ -invariant. By induction on  $l$ , the  $T^n$  action on  $L$  is effective.

The isotropy action of  $H_i$  on  $T_x M$  descends to the normal space  $N_x K_i$ . Up to a linear transformation,  $H_i$  acts on  $N_x K_i \cong \mathbb{C}$  by the  $U(1)$  action. Because the  $T^n$  action on  $L$  is effective, it follows that the  $T^n$ -invariant subspace  $T_x L \subset T_x M$  is transverse to  $T_x K_i$ . Therefore  $\widetilde{Z}$  is a normal crossing divisor in  $\widetilde{M}$  and  $D$  is a normal crossing divisor in  $\Delta$ .  $\square$

### Local normal form

Given a log symplectic manifold  $(M, Z, \omega)$  equipped with an effective  $T^n$  action, we show that intersections among the codimension 2 circle fixed point sets  $\{K_1, K_2, \dots, K_m\}$  are log symplectic submanifolds. We also show the existence of local normal forms for neighbourhoods of points on such submanifolds. We begin with a convenient definition of log symplectic submanifold.

**Definition A.15.** *Let  $(M, Z, \omega)$  be a log symplectic manifold. An embedded submanifold  $i : N \hookrightarrow M$  is called a log symplectic submanifold if  $\iota_* : TN \rightarrow TM$  is transverse to the anchor map  $TM(-\log Z) \rightarrow TM$  and  $\omega$  is nondegenerate on the resulting intersection.*

If  $Z$  is of normal crossing type, then the above transversality condition is equivalent to the condition that  $N$  be transverse to all possible intersections among components of  $Z$ . This means that  $Z_N = N \cap Z$  is a normal crossing divisor in  $N$ , and then  $N$  is log symplectic when  $\omega$  is nondegenerate when pulled back to the subbundle

$$TN(-\log Z_N) \hookrightarrow TM(-\log Z).$$



**Lemma A.16.** *Let  $(M, Z, \omega)$  be a  $2n$ -dimensional compact orientable log symplectic manifold with  $Z$  of normal crossing type, equipped with an effective  $T^n$  action. Let  $K$  be a codimension 2 component of the fixed point set  $M^H$  of a circle subgroup  $H \subset T^n$ .*

*Then  $K$  is a log symplectic submanifold. For every point  $x \in K$ , there exists a  $T^n$ -invariant neighbourhood  $U_x$  such that  $U_x$  is isomorphic to a neighbourhood of*

$$(x, 0) \in M^H \times \mathbb{C}. \quad (57)$$

*where  $\mathbb{C}$  is equipped with the symplectic structure*

$$\frac{i}{2} dz \wedge d\bar{z}$$

*and  $H$  acts on  $\mathbb{C}$  by the standard  $U(1)$  action.*

*Proof.* Let  $Z_1, Z_2, \dots, Z_l$  be components of  $Z$ . As shown in the proof of Lemma A.14,  $K$  is transverse to the intersection

$$\bigcap_{i=1, \dots, l} Z_i.$$

This implies  $K \cap Z$  is a normal crossing divisor in  $K$ . For every point  $x \in K$ , the log tangent space

$$T_x(M, -\log Z)$$

is a symplectic vector space, and the  $H$ -invariant subspace

$$T_x(K, -\log(K \cap Z))$$

is a symplectic subspace. This shows that  $K$  is a log symplectic submanifold.

Taking a smaller neighbourhood if necessary, we may assume that  $U_x$  is contractible and

$$U_x \cong (U_x \cap K) \times D$$

where  $D \subset \mathbb{C}$  is an open disk. We use the Moser method to show that  $(U_x, \omega)$  is equivariantly symplectomorphic to  $(U_x \cap K, \omega_K) \times (D, \sigma)$  where  $\omega_K$  is the induced log symplectic structure on  $U_x \cap K$  and  $D \subset \mathbb{C}$  is equipped with the symplectic structure  $\sigma = \frac{i}{2} dz \wedge d\bar{z}$  and the  $U(1)$  action. We write

$$\omega_t := (1 - t)\omega + t(\omega_K \oplus \sigma), \quad 0 \leq t \leq 1.$$

Because  $K \pitchfork (Z_i \cap Z_j)$ , for small enough  $U_x$ , we have

$$\bigoplus_{i,j} H^0(U_x \cap Z_i \cap Z_j) = \bigoplus_{i,j} H^0(K \cap U_x \cap Z_i \cap Z_j),$$

and therefore

$$H^2(U_x, \log Z) = H^2(K \cap U_x, \log(K \cap Z)).$$

This implies  $[\omega_0] = [\omega_t] \in H^2(U_x, \log(U_x \cap Z))$ , and so

$$\omega_0 - \omega_1 = td\tau'$$

for some  $\tau' \in \Omega^1(U_x, \log(U_x \cap Z))$ .

Choose a chart centered at  $x$  such that

$$\tau'|_x = \sum_{i=1}^l c_i \frac{dx_i}{x_i} + \sum_{i=l+1}^{2n} c_i dx_i. \quad (58)$$

We use the right hand side of (58) to define a closed logarithmic 1-form  $\tau_0$  on  $U_x$ . Consequently,

$$\omega_0 - \omega_1 = td\tau' = td(\tau' - \tau_0) = td\tau.$$

where  $\tau := \tau' - \tau_0$  has the property that  $\tau|_x = 0$ . We may average  $\tau$  over  $T^n$ , rendering  $\tau$   $T^n$ -equivariant. The usual Moser method yields the desired result.  $\square$

By Lemma A.16, we may deduce that  $K$  is a compact orientable log symplectic submanifold of dimension  $2n - 2$  with an induced effective action by the torus  $T^n/H$ , so we obtain the following result by induction on dimension.

**Proposition A.17.** *Let  $(M, Z, \omega)$  be a  $2n$ -dimensional compact orientable log symplectic manifold with  $Z$  of normal crossing type, equipped with an effective  $T^n$  action. Let  $\{K_1, \dots, K_m\}$  be the submaximal orbit type strata as described in A.12. Then*

$$N = K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_k}$$

*is a log symplectic submanifold with the induced action by the torus*

$$T^n / H_{i_1} H_{i_2} \dots H_{i_k}.$$

*For a point  $x \in N$ , there exists a  $T^n$ -invariant neighbourhood  $U_x$  that is equivariantly isomorphic to a neighbourhood of*

$$(x, 0) \in (N \cap U_p) \times \mathbb{C}^k. \quad (59)$$

*where  $\mathbb{C}^k$  is equipped with the symplectic structure*

$$\frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \dots + dz_k \wedge d\bar{z}_k)$$

*and each  $H_{i_p}$  acts only on the  $p$ -th factor of  $\mathbb{C}^k$  by the standard  $U(1)$  action.*

**Example A.18.** Let  $S^1$  act on

$$\left( \mathbb{C}, \sigma = \frac{i}{2} dz \wedge d\bar{z} \right)$$

by the standard  $U(1)$  action. Then, the blow-up of  $\mathbb{C}$  along 0

$$\text{Bl}_0(\mathbb{C}) \cong \left\{ (r, \theta) \mid r = |z| \in [0, \infty), \theta = \arg(z) \in S^1 \right\}$$

is endowed with the induced presymplectic structure  $rdr \wedge d\theta$ , and the compressed blow-up

$$\tilde{\mathbb{C}}_0 \cong \left\{ (t, \theta) \mid t = \frac{|z|^2}{2} \in [0, \infty), \theta = \arg(z) \in S^1 \right\}$$

is endowed with the induced symplectic structure  $\tilde{\sigma} = dt \wedge d\theta$ . The Hamiltonian function

$$t : \tilde{\mathbb{C}}_0 \rightarrow \mathbb{R}, \quad (t, \theta) \mapsto t$$

generates the induced  $S^1$  action on  $\tilde{\mathbb{C}}_0$ . The symplectic cut of  $(\tilde{\mathbb{C}}_0, \tilde{\sigma})$  with respect to the function  $t$  is equivariantly isomorphic to  $(\mathbb{C}, \sigma)$ .

### Symplectic uncut

We use Proposition A.17 to show that there is an induced  $T^n$ -invariant log symplectic structure on the compressed blow-up  $\tilde{M}$ . By Lemma A.14,  $\tilde{M}$  defines a log symplectic principal  $T^n$  bundle, which we call the symplectic uncut.

**Proposition A.19.** *Let  $(M, Z, \omega)$  be a  $2n$ -dimensional compact orientable log symplectic manifold with corners where  $Z$  is of normal crossing type, equipped with an effective  $T^n$  action. Let  $\Delta = M/T^n$  and  $D = Z/T^n$ .*

*Then the iterated compressed blow-up  $(\tilde{M}, \tilde{Z})$  in Lemma A.14, as a principal  $T^n$  bundle over  $(\Delta, D)$ , is endowed with a unique  $T^n$ -invariant log symplectic structure  $\tilde{\omega}$  such that the pullback of  $\tilde{\omega}$  to  $\text{Bl}_K(M)$  coincides with the pullback of  $\omega$  to  $\text{Bl}_K(M)$ , i.e. we have  $p^*\omega = q^*\tilde{\omega}$  in diagram 56.*

*Proof.* For a point

$$x \in N = K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_k} \subset M,$$

let  $U_x \subset M$  be a  $T^n$ -invariant neighbourhood that is isomorphic to a neighbourhood of

$$(x, 0) \in (N \cap U_x) \times \mathbb{C}^k.$$

as in Proposition A.17. In the proof of Lemma A.14, the preimage of  $q \circ p^{-1}(U_x) \subset \tilde{M}$  is equivariantly isomorphic to a neighbourhood of

$$(x, 0) \times T^k \subset (N \cap U_x) \times (\mathbb{R}_+ \times S^1)^k$$

where  $\mathbb{R}_+ = [0, \infty)$ . In these coordinates, the  $T^n$ -invariant symplectic form

$$\tilde{\omega} = \omega_N \oplus (dt_1 \wedge d\theta_1 + dt_2 \wedge d\theta_2 + \dots + dt_k \wedge d\theta_k)$$

satisfies  $p^*\omega = q^*\tilde{\omega}$ . □

**Definition A.20.** *Let  $(M, Z, \omega)$  be a compact orientable log symplectic manifold with corners with  $Z$  of normal crossing type, equipped with an effective  $T^n$  action. The triple  $(\tilde{M}, \tilde{Z}, \tilde{\omega})$ , as a log symplectic principal  $T^n$  bundle over  $(\Delta, D)$ , is called the symplectic uncut of  $(M, Z, \omega)$ .*

## A.4 Logarithmic de Rham cohomology

**A.21** Let  $(M, Z)$  be a manifold with real codimension 1 closed hypersurface. Let  $TM(-\log Z)$  be the log tangent algebroid. Then the residue exact sequence of de Rham complexes

$$0 \longrightarrow \Omega_M^k \longrightarrow \Omega_M^k(\log Z) \xrightarrow{\text{Res}} \Omega_Z^{k-1} \longrightarrow 0$$

may be smoothly split as a sequence of complexes, giving a natural decomposition of cohomology groups

$$H^k(M, \log Z) \cong H^k(M) \oplus H^{k-1}(Z), \quad (60)$$

recovering the well-known isomorphism of Mazzeo-Melrose [16].

**A.22** If  $Z$  is the degeneracy locus of a log symplectic structure, then  $Z$  has a foliation  $F$  by codimension 1 symplectic leaves, and the Poisson Lie algebroid is isomorphic to  $TM(-\log Z, F)$ , the sheaf of vector fields tangent to  $Z$  and to  $F$ . A local computation shows that the morphism of complexes dualizing the morphism of algebroids  $TM(-\log Z, F) \rightarrow TM(-\log Z)$ , namely

$$\Omega_M^\bullet(\log Z) \rightarrow \Omega_M^\bullet(\log Z, F),$$

is a quasi-isomorphism and so the Poisson cohomology coincides with the logarithmic cohomology of the hypersurface  $Z$ .

**A.23** In the case that there are several smooth hypersurfaces  $Z_1, \dots, Z_k$  which intersect in a normal crossing fashion, for example the boundary of a manifold with corners, the Mazzeo-Melrose theorem easily generalizes to give (for  $Z = Z_1 + \dots + Z_k$ )

$$H^k(M, \log Z) \cong H^k(M) \oplus \sum_i H^{k-1}(Z_i) \oplus \sum_{i < j} H^{k-2}(Z_i \cap Z_j) \oplus \dots \quad (61)$$

Furthermore, if  $Z$  is the degeneracy locus of a log symplectic structure, then the Poisson algebroid, as above, has de Rham complex quasi-isomorphic to the logarithmic de Rham cohomology of the hypersurface arrangement, allowing a computation of the Poisson cohomology by way of the Mazzeo-Melrose isomorphism.

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